FUZZY SETS AND THEIR APPLICATIONS

Objectives of the Course : To study about Fuzzy sets and their relations, Fuzzy graphs, Fuzzy relations, Fuzzy logic and laws of Fuzzy compositions .

Course Outline

UNIT-I : Fundamental Notions. Chapter I: Sec. 1 to 8

UNIT-II : Fuzzy Graphs. Chapter II: Sec. 10 to 18

UNIT-III : Fuzzy Relations. Chapter II: Sec. 19 to 29

UNIT-IV: Fuzzy Logic. Chapter III:Sec.31 to 40(omit Sec.37,38, 41)

UNIT-V: The Laws of Fuzzy Composition. Chapter IV: Sec.43 to 49

Recommended Text :

A.Kaufman, Introduction to the theory of Fuzzy subsets, Vol.I, Academic Press, New York, (1975).

Reference Books :

1. H.J.Zimmermann, Fuzzy Set Theory and its Applications, Allied Publishers, Chennai, (1996)

2. George J.Klir and Bo Yuan, Fuzzy sets and Fuzzy Logic-Theory and Applications, Prentice Hall India, New Delhi, (2001).

Course Learning Outcome (for Mapping with POs and PSOs)

Students will be able to

CLO1: Understand the definition of Fuzzy sets and its related concepts

CLO2: Define Fuzzy Graphs and can explain the concepts

CLO3: Explain the concepts in Fuzzy sets and its relations

CLO4: Discuss about Fuzzy logic

CLO5: Analyze the compositions of Fuzzy sets.

UNIT I

FUNDAMENTAL NOTIONS

1. INTRODUCTION

In this first chapter we review the principal definitions and concepts of the theory of ordinary sets, that is, those that are at the foundation of present-day mathematics; but these definitions and concepts will be reexamined and extended to notations that pertain to fuzzy subsets.

We shall progress rather slowly so that the reader who is not a mathematician but rather a user of mathematics will be able to follow without difficulty.

The examples will allow the reader to verify, step by step, whether the new notions have been well understood. But all that is presented in this first chapter is very simple; the difficulties will appear later.

The theory of ordinary sets is a particular ease of the theory of fuzzy subsets (we shall see presently why it is necessary to say fuzzy subset and not fuzzy set the reference set will not be fuzzy). We have here a new and very useful extension; but, as we shall note several times, what may be described or explained with the theory of fuzzy subsets may also be considered without this theory, using other concepts. One may always replace one mathematical concept with another. But will it be so clear or generative of properties that are easier to discover and prove, or to use?

2. REVIEW OF THE NOTION OF MEMBERSHIP

Let E be a set and A a subset of E

(2.1) $A \subset E$

One usually indicates that an element x of E is a member of A using the symbol $\in$

(2.2) $$x \in A$$

In order to indicate this membership one may also use another concept, a characteristic function $\mu_A(x)$, whose value indicates (yes or no) whether x is a member of A:

(2.3) $\mu_A(x) = 1 \;\; if x \in A$

$= 0\ if \quad x \notin A.$

Example . Consider a finite set with five elements:

(2.4) $E = \{x_1, x_2, x_3, x_4, x_5\}$

and let

(2.5) $A = \{x_2, x_3, x_5\}$

And one writes

(2.6) $\mu_A(x_1)$= 0, $\mu_A(x_2)$= 1, $\mu_A(x_3)$= 1, $\mu_A(x_4)$= 0, $\mu_A(x_5)$=1.

This allows us to represent A by accompanying the elements of E with their characteristic function values:

(2.7) $A = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1)\}$

Recall the well known properties of a Boolean binary algebra:

Let A^c be the complement of A with respect to E:

(2.8) $A \cap A^c = \emptyset,$

(2.9) $A \cup A^c = E.$

(2.10) If $x \in A, \quad x \notin A^c$, and one writes

(2.11) μ_A(x) = 1 and $\mu_{A^c}(x)$= 0

Considering the example in (2.6) and (2.7), one sees:

(2.12) $\mu_{A^c}(x_1) = 1, \mu_{A^c}(x_2) = 0, \mu_{A^c}(x_3) = 0, \mu_{A^c}(x_4) = 1, \mu_{A^c}(x_5) = 0,$

And one writes

(2.13) $A^c = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 1), (x_5, 0)\}$

Given now two subsets A and B, one may consider the intersection

(2.14) $A \cap B$

One has

(2.15) $\mu_A(\mathrm{x}) = 1 \ \ if x \in A$

$= 0 \ if \ \ \ x \notin A.$

(2.16) $\mu_B(\mathrm{x}) = 1 \ \ if x \in B$

$= 0 \ if \ \ \ x \notin B.$

(2.17) $\mu_{A\cap B}(\mathrm{x}) = 1 \ \ if x \in A \cap B$

$= 0 \ \ if \ \ \ x \notin A \cap B.$

This allows us to write

(2.18) $\mu_{A\cap B}(\mathrm{x}) = \mu_A(\mathrm{x}) . \mu_B(\mathrm{x})$,

Where the operation corresponds to the table in Figure 2.1 and is called the Boolean product.

(.)	0	1
0	0	0
1	0	1

FIG. 2.1

In the same fashion for the two subsets A and B , one defines the union or join:

(2.19) $\mu_{A\cup B}(\mathrm{x}) = 1 \ \ if x \in A \cup B$

$= 0 \ \ if \ \ \ x \notin A \cup B.$

With the property

(2.20) $\mu_{A\cup B}(\mathrm{x}) = \mu_A(\mathrm{x}) \dot{+} \mu_B(\mathrm{x})$

Where the operation $\dot{+}$, the Boolean sum, is defined by the table in figure 2.2.

($\dot{+}$)	0	1
0	0	1
1	1	1

FIG. 2.2

Example. Consider the reference set (2,4) and the two subsets

(2.21) $$A = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1)\},$$

(2.22) $$B = \{(x_1, 1), (x_2, 0), (x_3, 1), (x_4, 0), (x_5, 1)\},$$

One sets

(2.23) $$A \cap B = \{(x_1, 0.1), (x_2, 1.0), (x_3, 1.1), (x_4, 0.0), (x_5, 1.1)\},$$

$$= \{(x_1, 0), (x_2, 0), (x_3, 1), (x_4, 0), (x_5, 1)\}$$

(2.24) $$A \cup B = \{(x_1, 0\dotplus 1), (x_2, 1\dotplus, 0), (x_3, 1\dotplus 1), (x_4, 0\dotplus 0), (x_5, 1\dotplus 1)\},$$

$$= \{(x_1, 1), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1)\},$$

To continue, emanating from these two subsets one has

(2.25) $$\overline{A \cap B} = \{(x_1, 1), (x_2, 1), (x_3, 0), (x_4, 1), (x_5, 0)\},$$

(2.26) $$\overline{A \cup B} = \{(x_1, 0), (x_2, 0), (x_3, 0), (x_4, 1), (x_5, 0)\},$$

These few exercise constitute only a didactic preamble to an understanding of fuzzy subsets.

3. THE CONCEPT OF A FUZZY SUBSET

We shall begin with an example. Consider the subset A of E defined by (2.7) The five elements of E belong or do not belong to A, one or the other. The characteristic function takes only the values 0 or 1.

Imagine now that this characteristic function may take any value whatsoever in the interval [0, 1]. Thus, an element x_i of E may not be a member of A(μ_A= 0) could be a member of A a little ($\mu_A$near 0), may more or less be a member of A (μ_A neither too near 0 nor too near 1), could be strongly a member of A ($\mu_A$near 1), or finally might be a member of A(μ_A = 1). In this manner the notion of membership takes on an interesting extension and leads, as we shall see, to very useful developments.

The mathematical concept is defined by the expression

(3.1) $$\underset{\sim}{A} = \{(x_1|0.2), (x_2|0), (x_3|0.3), (x_4|1), (x_5|0.8)\}$$

Where x_i is an element of the reference set E and where the number placed after the bart is the value of the characteristic function for the element; this mathematical concept will be called a fuzzy subset of E and be denoted

(3.2) $\underset{\sim}{A} \subset E$

One may denote membership in a fuzzy subset by

(3.3) $x \underset{0.2}{\in} \underset{\sim}{A}$, $y \underset{1}{\in} \underset{\sim}{A}$, $z \underset{0}{\in} \underset{\sim}{A}$,

The symbol $\underset{1}{\in}$ may be taken to be equivalent to $\in$, and $\underset{0}{\in}$ to $\notin$. In order to avoid encumbering the notation, one uses simply $\in$ to indicate membership and $\notin$, nonmembership.

Thus, the fuzzy subset defined by (3, 1) contains a littlex_1 does not contain x_2 . contains a little more x_3 ,contains x_4 completely, and a large part of x_5 This will allow us to construct a mathematical structure with which one may be able to manipulate con- cepts that are rather poorly defined but for which membership in a subset is somewhat hierarchical. Thus, one may consider: in the set of men, the fuzzy subset of very tall men; in the set of basic colors, the fuzzy subset of deep green colors; in the set of decisions, a fuzzy subset of good decisions and so forth. We shall go on to see how to manipulate these concepts that seem particularly well adapted to the imprecision prevalent in the social sciences.

Let E be a set, denumerable or not, and let x be an element of E. Then a fuzzy subset $\underset{\sim}{A}$ *of E is a set of ordered pairs*

$$\left\{\left(x, \mu_{\underset{\sim}{A}}(x)\right)\right\} \; \forall\, x \in E.$$

where $\mu_{\underset{\sim}{A}}(x)$ *is a membership characteristic function that takes its values in a totally ordered set M, and which indicates the degree or level or membership. M will be called a membership set.*

A vertical bar has been used in place of a comma, as in (2.7), in order to avoid confusion. When one is using the American decimal point, a comma may, of course, be used in place of the bar.

We have, however, adapted this definition to the terminology and presentation of the present work.

A set or a subset is denoted in the present work by a boldface letter: A, X, a, p, ……. A fuzzy subset will be designated by a boldface letter under which is placed the symbol ∼.

Thus

$$\underset{\sim}{A},\quad \underset{\sim}{X},\quad \underset{\sim}{a},\quad \underset{\sim}{p}$$

represent fuzzy subsets. When all the subsets turn out to be ordinary, one may, if it is useful, suppress the small supplementary symbol ~ .

Membership and nonmembership will be indicated by

(3.8) $\in$ and $\notin$;

fuzzy membership and fuzzy nonmembership will be represented by

(3.9) $\underset{\sim}{\in},\quad \underset{\sim}{\notin},$ if this is necessary.

In certain cases where the totally ordered set M, in which $\mu_A(x)$ takes its values, is the doubly closed interval [0, 1], it may be convenient to accompany the symbol E by a number from [0, 1] placed beneath it. Thus

(3.10) $x \underset{1}{\in} \underset{\sim}{A}$, indicates $x \in A$, that is, "x is a member of A,"

$x \underset{0}{\in} \underset{\sim}{A}$, indicates $x \notin A$, that is, "x is not a member of A,"

$x \underset{0.8}{\in} \underset{\sim}{A}$, indicates that x is a member of A with degree 0,8,

and so forth. The next examples will be very useful.

Example 1. Consider a finite set:

(3.11) E= {a, b, c, d, e, f}

and the finite ordered set

M=(0, 1/2, 1).

(3.12)

Then

(3.13) $\underset{\sim}{A}$ = ((a|0), (b|1), (c|$\frac{1}{2}$). (d|0), (e|$\frac{1}{2}$),(f|0))

is a fuzzy subset of E and one may write

$$a \underset{0}{\in} \underset{\sim}{A} \quad , \ \mathrm{b} \underset{1}{\in} \underset{\sim}{A} \quad , \qquad \mathrm{c} \underset{1/2}{\in} \underset{\sim}{A} \quad , \ \mathrm{d} \underset{0}{\in} \underset{\sim}{A} \quad , etc.$$

Example 2. Let N be the set of natural numbers:

(3.14) N=(0,1,2,3,4,5,6,...)

and consider the fuzzy subset $\underset{\sim}{A}$ of "small" natural numbers:

(3.15) $\underset{\sim}{A}$ = ((0|1), (1|0,8), (2|0,6), (3|0,4), (4|0,2), (5|0), (6|0).....

Here, of course, the functional values $\mu_A(x)$, where x = 0, 1, 2, 3,..., have been given subjectively. One may write

(3.16) $0 \underset{1}{\in} \underset{\sim}{A} \quad , 1 \underset{0.8}{\in} \underset{\sim}{A} \quad , 2 \underset{0.6}{\in} \underset{\sim}{A} \quad , 3 \underset{0.4}{\in} \underset{\sim}{A} \quad , \ldots \ldots \ldots \ldots \ldots ..$

Example 3. Let E be the finite set of the first ten integers:

(3.17) E=(0,1,2,3,4,5,6,7,8,9)

and consider the fuzzy subset $\underset{\sim}{A}$ containing the numbers of E in the following fashion:

$\underset{\sim}{A}$ ={(0|0), (1|0,2), (2|0,3), (3|0), (4|1).

(3.18) (5|1). (6|0,8). (7|0,5), (8|0), (9|0)),

where again the $\mu_{\underset{\sim}{A}}(x)$ are subjective.

One may write

(3.19)) $0 \underset{0}{\in} \underset{\sim}{A} \quad , 1 \underset{0.2}{\in} \underset{\sim}{A} \quad , 2 \underset{0.3}{\in} \underset{\sim}{A} \quad , 3 \underset{0}{\in} \underset{\sim}{A} \quad , \ldots \ldots \ldots \ldots \ldots ..$

The reader will note that this symbol of generalized membership may be employed in the opposite sense. Thus, for (3.13) one may write

(3.20) $\underset{\sim}{A} \underset{0}{\ni} a, \ \underset{\sim}{A} \underset{1}{\ni} b, \ \ \underset{\sim}{A} \underset{0.5}{\ni} c.$

and for (3.19),

(3.21) $\underset{\sim}{A} \underset{0}{\ni} 0, \ \underset{\sim}{A} \underset{0.2}{\ni} 1, \ \ \underset{\sim}{A} \underset{0.3}{\ni} 2.$

4. DOMINANCE RELATIONS

Recall first the nature of a dominance relation existing between two ordered n-tuples. Consider the two ordered n-tuples.

(4.1) $v = \{k_1, k_2, \ldots, k_n\}$

And

(4.2) $v' = \{k'_1, k'_2, \ldots, k'_n\}$

in which the k_i and the k'_i, i = 1, 2,..., n, belong to the same totally ordered set K, in which the relation of order will be represented by the symbol >.

We shall say that v' dominates r, which is written

(4.3) $v' \geq r.$

if and only if

(4,4) $k'_1 > k_1,\ k'_2 > k_2, \ldots\ldots\ldots\ldots\ldots\ldots k'_n > k_n,$

The symbols ≥ and >for the order relation correspond to a nonstrict order relation. If we then use the symbols > and ⋗ corresponding to a strict order relation, we say that v'strictly dominates v. One may then see that

(4.5) $v' \gtrdot v$

if and only if

(4.6) $k'_1 > k_1,\ k'_2 > k_2, \ldots\ldots\ldots\ldots\ldots\ldots k'_n > k_n,$

with atleast one k'_i and one k_i between which there exist a strict relation.

Granting what has been developed here, one may then say that all dominance relations introduce an order relationship (total or partial) between n-tubles such as v and v'.

Example 1. Consider the four tuples

(4.7) $u = (7,3,0,5),$

(4.8) $v = (2,2,0,4),$

(4.9) $w = (3,4,1,4),$

One sees that

(4.10) $u > v$ since $7 > 2, 3 > 2, 0 = 0, 5 > 4,$

Since one of the terms of u, atleast, is greater than the corresponding term of v, one may likewise write $u > v$. In the same manner one may verify that $w > v$. But u and w are not comparable. In fact,

(4.11) $7 > 3, 3 < 4, 0 < 1, 5 > 4.$

Example 2.

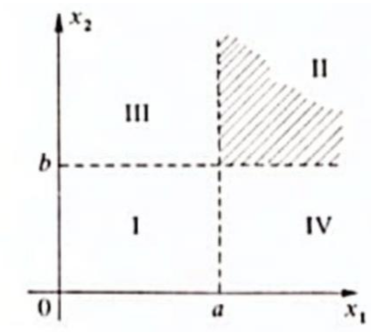


Figure 4.1

Consider the set P of points (x_1, x_2) in the plane indicated in Figure 4.1 and defined by $x_1 \geq 0$ and $x_2 \geq 0$. All points of the shaded domain II, that is, those with $x_1 \geq a$, $x_2 \geq b$, dominate and in fact strictly dominate all points of domain I, $0 \leq x_2 < a$, $0 \leq x_2 < b$. All points of domain III are not necessarily comparable with all points of IV on the one hand with I and II on the other. Finally, each point of III is not comparable to a point of IV and vice versa, evidently, except those points such that $x_1 = a$ or $x_2 = b$.

5.SIMPLE OPERATIONS ON FUZZY SUBSETS

Inclusion. Let E be a set and M its associated membership set, and let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two fuzzy subsets of E; we say that $\underset{\sim}{A}$ is included in $\underset{\sim}{B}$ if

(5.1) $\forall\, x \epsilon E \quad : \mu_{\underset{\sim}{A}}(x) \leq \mu_{\underset{\sim}{B}}(x)$

This will be denoted by

(5.2) $\underset{\sim}{A} \subset \underset{\sim}{B}$

And, if necessary to avoid confusion,

(5.3) $\underset{\sim}{A} \underset{\sim}{\subset} \underset{\sim}{B}$

which says very precisely that it is a case of inclusion in the sense of the theory of fuzzy subsets.

Strict inclusion, corresponding to the case where at least one relation in (5.1) is strict, will be denoted

(5.4) $\underset{\sim}{A} \subset\subset \underset{\sim}{B}$ or $\underset{\sim}{A} \underset{\sim}{\subset} \underset{\sim}{\subset} \underset{\sim}{B}$

We will consider three examples

(1) Let

(5.5) $E = \{x_1, x_2, x_3, x_4\}, M = [0,1]$

(5.6) $\underset{\sim}{A} = \{(x_1|0.4),\ (x_2|0.2),\ (x_3|0),\ (x_4|1)\},$

(5.7) $\underset{\sim}{B} = \{(x_1|0.3),\ (x_2|0),\ (x_3|0),\ (x_4|0)\},$

One has

(5.8) $\underset{\sim}{B} \subset \underset{\sim}{A}$ since $0.3 < 0.4, 0 < 0.2, 0 = 0, 0 < 1$

(2) Let

(5.9) $\underset{\sim}{A} \subset E$, $\underset{\sim}{B} \subset E$ $M = [0,1]$

If

(5.10) $\forall x \in E\ : \mu^2_{\underset{\sim}{A}}(\mathrm{x}) = \mu_{\underset{\sim}{B}}(\mathrm{x})$

Then

(5.11) $\underset{\sim}{B} \subset \underset{\sim}{A}$

(3) Let

$$E = \{x_1, x_2, x_3, x_4, x_5\}, M = [0,1]$$

One can write

(5.12) $E = \{(x_1, 1), (x_2, 1), (x_3, 1), (x_4, 1), (x_5, 1)\}$

Thus E is also included in itself in the sense of the theory of fuzzy subsets:

(5.13) $\underset{\sim}{E} \subseteq$ E

And this property remains true whatever the set E may be.

Equality. Let E be a set and M its associated membership set, and let and be two fuzzy subsets of E; we say that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are equal if and only if

(5.14) $\forall x \in E;\ \mu_{\underset{\sim}{A}}$ (x) = $\mu_{\underset{\sim}{B}}$ (x)

This will be denoted by

(5.15) $\underset{\sim}{A} = \underset{\sim}{B}$

If at least one of x of E is such that the equality is not satisfied , we say that and are not equal, and this will be denoted

(5.16) $\underset{\sim}{A} \neq \underset{\sim}{B}$

Complementation. Let E be a set and $M = [0,1]$ its associated membership set and let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two fuzzy subsets of E; we say that $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are complementary if

(5.17) $\forall x \in E;\ \mu_{\underset{\sim}{B}}$ (x) = 1 - $\mu_{\underset{\sim}{A}}$ (x)

This will be denoted

(5.18) $\underset{\sim}{B} = \underset{\sim}{\bar{A}}$ or $\underset{\sim}{\bar{A}} = \underset{\sim}{B}$

One obviously always has

(5.19) $(\underset{\sim}{\bar{A}}) = \underset{\sim}{A}$

We note that here complementation is defined for $M = [0,1]$, but one may extend this to other ordered membership sets M using other appropriate definitions.

We consider an example

(5.20) $E = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $M = [0,1]$.

(5.21) $\underset{\sim}{A} = \{(x_1, |0.13), (x_2, |0.61), (x_3, |0), (x_4, |0)(x_5, |1), (x_6, |0.03)\}$

$\underset{\sim}{B} = \{(x_1, |0.87), (x_2, |0.39), (x_3, |1), (x_4, |1)(x_5, |0), (x_6, |0.97)\}$.

Then certainly

(5.22) $\underset{\sim}{\bar{A}} = \underset{\sim}{B}$

Intersection. Let E be a set and $M = [0,1]$ its associated membership set, and let $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ be two fuzzy subsets of E ; one defines the intersection

(5.23) $\underset{\sim}{A} \cap \underset{\sim}{B}$

as the largest fuzzy subset contained at the same time in A and B. That is,

(5.24) $\forall\ x \in E : \mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \text{MIN}\ (\mu_{\underset{\sim}{A}}(x)\ ,\ \mu_{\underset{\sim}{B}}(x))$

Example

(5.25) $E = \{x_1\ ,\ x_2\ ,\ x_3\ , x_4\ ,\ x_5\}\ ,\ M = |0\ ,\ 1|$

(5.26) $\underset{\sim}{A} = \{\ (x_1 \mid 0.2),\ (x_2 \mid 0.7), (x_3 \mid 1),\ (x_4 \mid 0),\ (x_5 \mid 0.5)\ \}.$

(5.27) $\underset{\sim}{B} = \{\ (x_1 \mid 0.5),\ (x_2 \mid 0.3), (x_3 \mid 1),\ (x_4 \mid 0.1),\ (x_5 \mid 0.5)\ \}.$

(5.28) $\underset{\sim}{A} \cap \underset{\sim}{B} = \{\ (x_1 \mid 0.2),\ (x_2 \mid 0.3), (x_3 \mid 1),\ (x_4 \mid 0),\ (x_5 \mid 0.5)\ \}.$

Referring to the general definition (5.23) and (5.24), one may, moreover, write

(5.29) $\forall\ x \in E : x \underset{\mu_{\underset{\sim}{A}}}{\in} A$ and $x \underset{\mu_{\underset{\sim}{B}}}{\in} \mathbf{B}$ - $x \underset{\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}}{\in} \underset{\sim}{A} \cap \underset{\sim}{B}$

This permits us to introduce a fuzzy and to be symbolized and.

Thus one may say: If $\underset{\sim}{A}$ is the fuzzy subset of real numbers very near 5 and $\underset{\sim}{B}$ the fuzzy subset of real numbers very near 10, then $\underset{\sim}{A} \cap \underset{\sim}{B}$ is the fuzzy subset of real numbers very near to 5 and to 10. The fuzzy conjunction and is pronounced as and, but except where necessary, one may omit placing ~ beneath it.

Union. Let E be a set and M [0, 1] its associated membership set, and let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two fuzzy subsets of E; we define the union.

(5.30) $\underset{\sim}{A} \cup \underset{\sim}{B}$

as the smallest fuzzy subset that contains both $\underset{\sim}{A}$ and $\underset{\sim}{\boldsymbol{B}}$. That is,

(5.31) $\forall\ x \in E : \mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \text{MAX}\ (\mu_{\underset{\sim}{A}}(x)\ ,\ \mu_{\underset{\sim}{B}}(x))$

Recalling the example presented in (5.25)-(5.27), one sees

(5.32) $\underset{\sim}{A} \cup \underset{\sim}{B} = \{\ (x_1 \mid 0.5),\ (x_2 \mid 0.7), (x_3 \mid 1),\ (x_4 \mid 0),\ (x_5 \mid 0.5)\ \}.$

And recalling the general definitions (5,30), (5.31), one may, moreover, write

(5.33) $\forall\ x \in E : x \underset{\mu_{\underset{\sim}{A}}}{\in} A$ or/and $x \underset{\mu_{\underset{\sim}{B}}}{\in} \mathbf{B}$ - $x \underset{\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}}{\in} \underset{\sim}{A} \cup \underset{\sim}{B}$

This allows us to introduce a fuzzy or/and, to be symbolized or/and. Except where necessary, one omits the symbol ~.

Thus one may say: If $\underset{\sim}{A}$ is the fuzzy subset of real numbers very near 5 and $\underset{\sim}{B}$ the fuzzy subset of real numbers very near 10, then $\underset{\sim}{A} \cup \underset{\sim}{B}$ is the fuzzy subset of real numbers very near to 5 or/and to 10. The conjunction or/and is pronounced as or / and.

Remark. When there is no possibility of error in interpretation, one will write "and" for "and," and in the same manner "or/and" for "or/and".

Disjunctive sum. The disjunctive sum of two fuzzy subsets is defined in terms of unions and intersections in the following fashion:

(5.34) $\underset{\sim}{A} \oplus \underset{\sim}{B} = (\underset{\sim}{A} \cap \underset{\sim}{\bar{B}}) \cup (\underset{\sim}{\bar{A}} \cap \underset{\sim}{B})$

This operation corresponds to "fuzzy disjunctive or," where "or" is read "or" and will be written "or" when there is no risk of error.

We consider an example (the example that has served for union and intersection).

(5.35) $\underset{\sim}{A}$ = { (x_1 | 0.2), (x_2 | 0.7),(x_3 |1), (x_4 |0), (x_5 |0.5) }.

(5.36) $\underset{\sim}{\boldsymbol{B}}$ = { (x_1 | 0.5), (x_2 | 0.3),(x_3 |1), (x_4 |0.1), (x_5 |0.5) }.

(5.37) $\underset{\sim}{\bar{A}}$ = { (x_1 | 0.8), (x_2 | 0.3),(x_3 |0), (x_4 |0.1), (x_5 |0.5) }.

(5.38) $\underset{\sim}{\bar{B}}$ = { (x_1 | 0.5), (x_2 | 0.7),(x_3 |0), (x_4 |0.9), (x_5 |0.5) }.

(5.39) $\underset{\sim}{A} \cap \underset{\sim}{\bar{B}}$ = { (x_1 | 0.2), (x_2 | 0.7),(x_3 |0), (x_4 |0), (x_5 |0.5) }.

(5.40) $\underset{\sim}{\bar{A}} \cap \underset{\sim}{B}$ = { (x_1 | 0.5), (x_2 | 0.3),(x_3 |0), (x_4 |0.1), (x_5 |0.5) }.

(5.41) $\underset{\sim}{A} \oplus \underset{\sim}{B}$ = { (x_1 | 0.5), (x_2 | 0.7),(x_3 |0), (x_4 |1), (x_5 |0.5) }.

Difference. The difference is defined by the relation

(5.42) $\underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{A} \cap \underset{\sim}{\bar{B}}$

Considering again the example (5.26) and (5.27), and using (5.38) and (5.39), (5.43) $\underset{\sim}{A} \cap \underset{\sim}{\bar{B}}$ = { (x_1 | 0.2), (x_2 | 0.7),(x_3 |0), (x_4 |0), (x_5 |0.5) }.

Of course, except in particular cases,

(5.44) $\underset{\sim}{A} - \underset{\sim}{B} \neq \underset{\sim}{B} - \underset{\sim}{A}$

Visual representation of simple operations on fuzzy subsets. For fuzzy subsets, one may construct a visual representation allied with that for ordinary subsets (Venn-Euler diagrams).

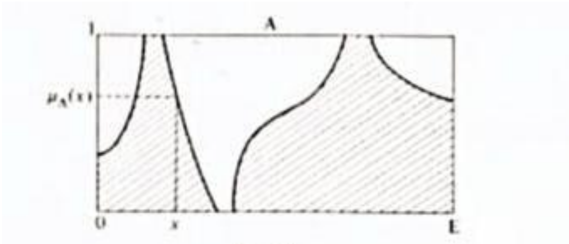
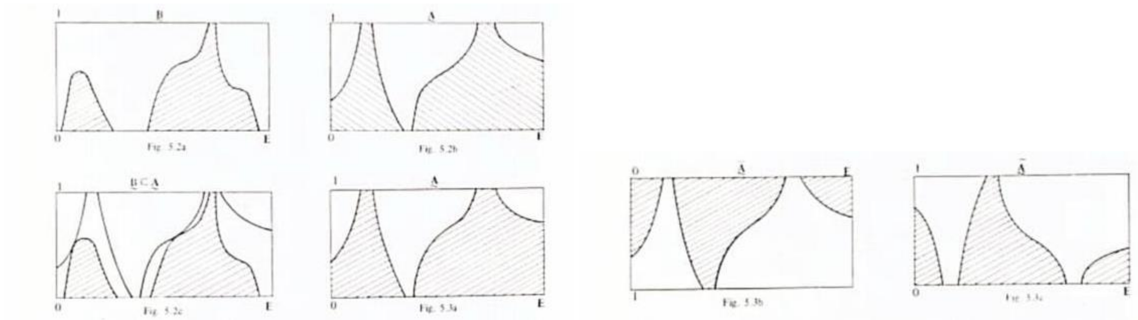


Fig. 5.1

Consider a rectangle (Figure 5.1) with the values ofμ_A (x) as ordinate and as an abscissa the elements of E in an arbitrary order (if there is in the nature of E a total order, that order will be taken). In Figure 5.1 the membership of each element is represented by its ordinate. The shaded part conveniently represents the fuzzy subset $\underset{\sim}{A} \subset$ E.

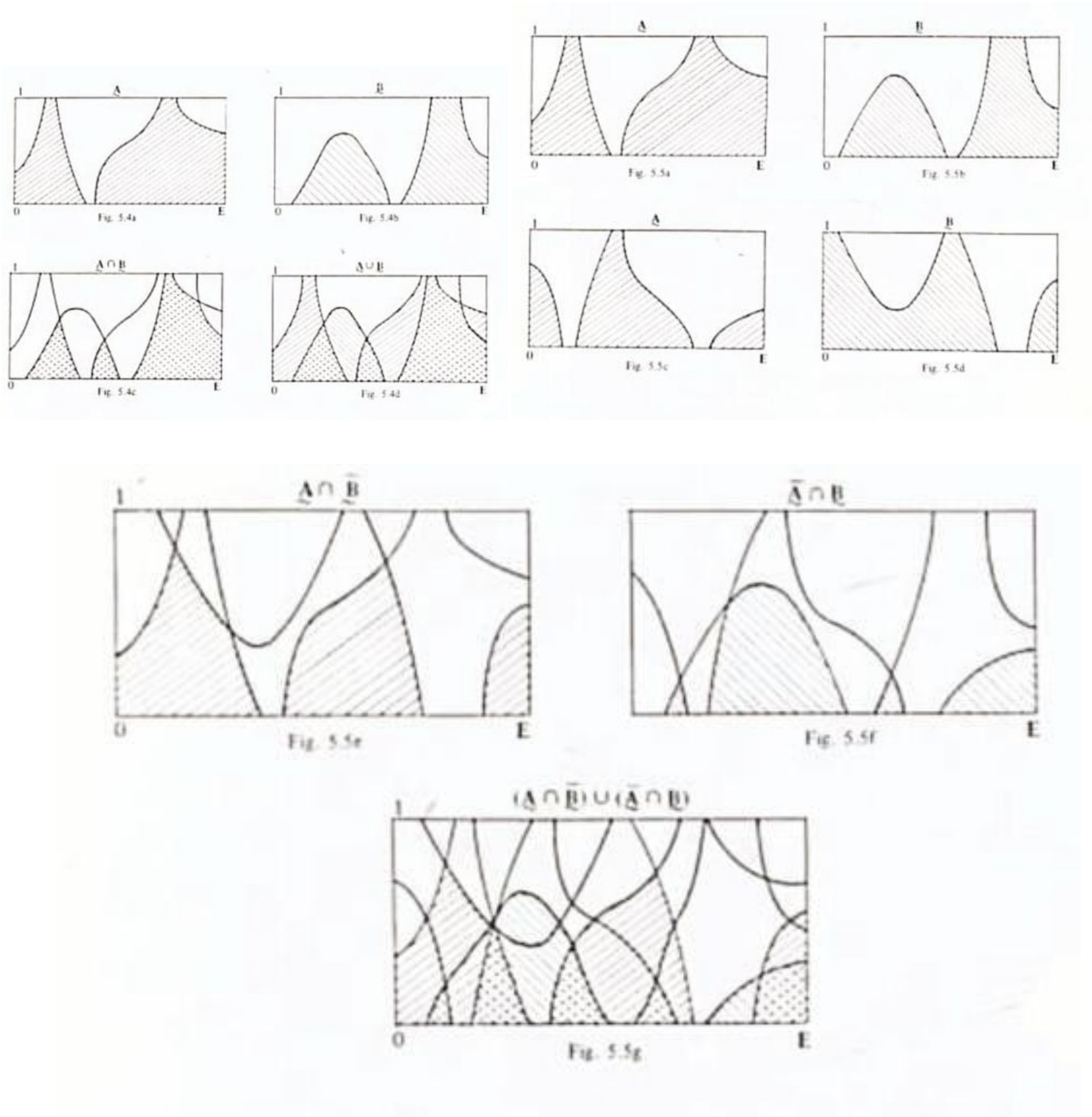
With this representation we see how to visualize the various simple operations on fuzzy subsets. A series of figures will show how to use this representation.

In Figures 5.2-c the property of inclusion is presented. Figures 5.3a-c illustrate complementation. The properties of union and intersection are shown in Figures 5.42-d.



This represents the fuzzy subset $\underset{\sim}{A}$ and contains all the fuzzy subsets that are included in $\underset{\sim}{A}$. These shadings are convenient for distinguishing one fuzzy subset from another.

In Figures 5.5a-g are represented the properties of the difference $\underset{\sim}{A} - \underset{\sim}{B} = \underset{\sim}{A} \cap \underset{\sim}{\bar{B}}$ and the disjunctive sum $\underset{\sim}{A} \oplus \underset{\sim}{B} = (\underset{\sim}{A} \cap \underset{\sim}{\bar{B}}) \cup \left(\underset{\sim}{\bar{A}} \cap \underset{\sim}{B}\right)$.



Hamming distance. We recall first what is meant by Hamming distance in the theory of ordinary subsets. Consider two ordinary subsets A ⊂ E, B ⊂ E. E finite.

(5.45)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
A =	1	0	0	1	0	1	0

(5.46)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
B =	0	1	0	0	0	1	1

The Hamming distance between A and B is the quantity

(5.47) $d(A,B) = \sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))$

For the example in (5.45) and (5,46), one has d(A, B) = |1 - 0| + |0 - 1| + |0 - 0| + |1 - 0| + |0 - 0| +|1 – 1| + |0 - 1|

(5.48) = 1+1+0+1+0+0+1 = 4.

The reader knows that the word distance may not be used arbitrarily in mathematics. If X and Y are two elements between which one wishes to define a distance, it is necessary. you will recall, that one have, for some operation.:

∀ X,Y,Z∈ E:

(5.49) 1) d(X, Y)> 0.Nonnegativity

(5.50) 2) d(X, Y) = d(Y, X), symmetry

(5.51) 3) d(x, z) < d(X, Y). d(Y, Z).

transitivity for the operation associated with the notion of distance.

To these three conditions, one may add a fourth:

(15.52) 4) d(X, X)= 0.

One may easily verify that a lamming distance is indeed a distance in the sense given by (5.49)-(5.52) with the operation+ (ordinary sum).

We define also, for a finite E with n card E-4the number of elements in E), a relative lamming distance:

(5.53) $\delta\ (A\ ,B) = \frac{1}{n}$d(A , B)

For the example in (5.45) and (5,46), one has

$\delta\ (A\ ,B)\ = \frac{d\ (A\ ,B)}{7}\ = \ \frac{4}{7}$

One has always

(5.54) 0<$\delta\ (A\ ,B)$< 1.

With a view toward generalizing the notion of Hamming distance to the case where one considers fuzzy subsets and not only ordinary subsets, we state two theorems.

Theorem I. Let p_i ,m_i, n_i ϵ R^* , i = 1 , 2, … , k; then

(5.55) $(p_i < m_i + n_i.\quad i = (1,2,\dots\dots\dots.k)\ )\qquad \sum_{i=1}^{k} p_i < \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} n_i$

Proof. This result is immediate upon forming the sums from: 1 to k on the left and right sides of the inequality.

Theorem II. Let p_i, m_i, $n_i \in R^*$, i = 1 , 2, … , k; then

(5.56) $(p_i < m_i + n_i. \quad i = (1,2, \ldots \ldots \ldots . k))$ $\sqrt{\sum_{i=1}^{k} p_i^2} < \sqrt{\sum_{i=1}^{k} m_i^2} + \sqrt{\sum_{i=1}^{k} n_i^2}$

Proof. This result is less immediate.

One may give another proof based on the theory of complex numbers; we have preferred a direct presentation.

We treat the evident inequality

(5.57) $\sum_{i=1}^{k} (m_i n_i - m_i n_i)^2 > 0$

Developing this sum of squares, we have

(5.58) $\sum_{i=1}^{k} m_i^2 n_i^2 - 2\sum_{i=1}^{k} m_i\ n_i\ m_j n_j > 0.$

That is,

(5.59) $\sum_{i=1}^{k} m_i^2 n_i^2 > 2 \sum_{i=1}^{k} m_i\ n_i\ m_j n_j$

Adding $\sum_{i=1}^{k} m_i^2 n_i^2$ to the two members of this inequality,

(5.60) $\sum_{i=1}^{k} m_i^2 n_i^2 + \sum_{i=1}^{k} m_i^2 n_i^2 > \sum_{i=1}^{k} m_i^2 n_i^2 + \sum_{i=1}^{k} 2m_i\ n_i\ m_j n_j$

which may be rewritten

(5.61) $(\sum_{i=1}^{k} m_i^2)(\sum_{i=1}^{k} n_i^2) > (\sum_{i=1}^{k} m_i n_i)^2$

(562) $\sqrt{\sum_{i=1}^{k} m_i^2}\ \sqrt{\sum_{i=1}^{k} n_i^2} > \sum_{i=1}^{k} m_i n_i$

(5.63) $2\ \sqrt{\sum_{i=1}^{k} m_i^2}\ \sqrt{\sum_{i=1}^{k} n_i^2} > 2\ \sum_{i=1}^{k} m_i n_i$

Adding, $\sum_{i=1}^{k} m_i^2 + \sum_{i=1}^{k} n_i^2$ one has

(5.64) $\sum_{i=1}^{k} m_i^2 + \sum_{i=1}^{k} n_i^2 + 2\sqrt{\sum_{i=1}^{k} m_i^2}\ \sqrt{\sum_{i=1}^{k} n_i^2} > \sum_{i=1}^{k} m_i^2 + \sum_{i=1}^{k} n_i^2 + 2\ \sum_{i=1}^{k} m_i n_i$

which may be rewritten

(5.65) $(\sqrt{\sum_{i=1}^{k} m_i^2} + \sqrt{\sum_{i=1}^{k} n_i^2})^2 > \sum_{i=1}^{k} (m_i + n_i)^2$

(5.66) $\sqrt{\sum_{i=1}^{k} m_i^2} + \sqrt{\sum_{i=1}^{k} n_i^2} > \sqrt{\sum_{i=1}^{k} (m_i + n_i)^2}$

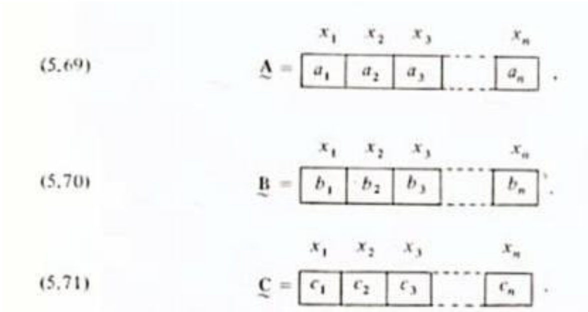
But, by hypothesis,

(5.67) $\forall_i$=1,2,.......k: $m_i + n_i > p_i$.

And then

(5.68) $\sqrt{\sum_{i=1}^{k} m_i^2} + \sqrt{\sum_{i=1}^{k} n_i^2} > \sqrt{\sum_{i=1}^{k} p_i^2}$

Generalization of the notion of Hamming distance. Consider now three fuzzy sub- sets $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C} \subset$ E, E finite, card E=n:



Suppose that one has defined a distance, denoted $\omega(a_i,b_i)$, between a_i and b_i for all i= 1, 2,...,n, and that the same holds for (b_i , c_i) and for (a_i, c_i). One must then have, since it is a distance according to (5.49)-(5.52).

(5.72) $\forall$ = 1,2,3 .... n$\omega(a_i,c_i) < \omega(a_i,b_i) = \omega(b_i,c_i)$

And, according to Theorems I (5.55) and 11 (5.56), one may write

(5.73) $\sum_{i=1}^{n} \omega(ai\,,ci) < \sum_{i=1}^{n} \omega(ai\,,bi) + \sum_{i=1}^{n} \omega(bi\,,ci)$

(5.74) $\sqrt{\sum_{i=1}^{n} \omega^2(a_i\,,c_i)} < \sqrt{\sum_{i=1}^{n} \omega^2(a_i\,,b_i)} + \sqrt{\sum_{i=1}^{n} \omega^2(b_i\,,c_i)}$

These two formulas give two evaluations of the distance between fuzzy subsets, one linear and the other quadratic.

Now we consider the case where, in fuzzy subsets, the membership function takes its values in M = [0 ,1] that is, where one has in (5.69)-(5.71), a_i , b_i , $c_i \subset$ [0 ,1] ,i = 1, 2,...,n.

Now take

(5.75) $\omega(a_i,b_i) = |a_i - b_i|$, $\omega(b_i,c_i) = |b_i - c_i|$, $\omega(a_i,c_i) = |a_i - c_i|$

The notion of distance has been the subject of a number of works. We present here two notions among those used most often. Of course, one may define other notions of distance for fuzzy subsets and we define two types of distance corresponding to (5.73) and (5.74):

Generalized Hamming distance or linear distance. This will be defined by

(5.76) $\quad$ d($\underset{\sim}{A}$, $\underset{\sim}{B}$, B)= $\sum_{i=1}^{n} |\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i)(x_i)|$

This generalizes (5.47) to the case where

(5.76a) $\quad \mu_{\underset{\sim}{A}}(x_i)$, $\mu_{\underset{\sim}{B}}(x_i) \in [0 , 1]$, i = 1 , 2,…. n .

And one has

(5.77) $\quad 0<d(\underset{\sim}{A} , \underset{\sim}{B}) < n$

Euclidean distance or quadratic distance. This will be defined by

(5.78) $e(\underset{\sim}{A} , \underset{\sim}{\boldsymbol{B}}) = \sqrt{\sum_{i=1}^{n}(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i))^2}$

One has

(5.79) $\quad 0 < e(\underset{\sim}{A} , \underset{\sim}{B}) <\sqrt{n}$

The quantity $e^2(A, B)$ is called the euclidean norm:

(5.80) $\quad e^2(\underset{\sim}{A} , \underset{\sim}{B}) = \sum_{i=1}^{n}(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i))^2$

We now define some relative distances.

Generalized relative Hamming distance

(5.81) $\quad \delta(\underset{\sim}{A} , \underset{\sim}{B}) = \frac{d(A,B)}{n} = \frac{1}{n}\sum_{i=1}^{n} |\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i)|$

One may verify that this is indeed a distance according to (5.49)-(5.52), and with reference to (5.73), where the property has not been altered by dividing the two members by n, one has

(5.82) $\quad 0< \delta(\underset{\sim}{A} , \underset{\sim}{B}) < 1.$

and (5.81) generalizes (5.53) for the case where $\mu_{\underset{\sim}{A}}(x_i), \mu_{\underset{\sim}{B}}(x_i) \in [0 , 1]$

Relative euclidean distance

(5.83) $\quad e(\underset{\sim}{A} , \underset{\sim}{B}) = \frac{e(\underset{\sim}{A} , \underset{\sim}{B})}{\sqrt{n}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i))^2}$

One may verify that this is indeed a distance according to (5.49)-(5.52), and with reference to (5.74), where the property is not altered by dividing the two members by √n.

one has

Note that we have $|\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i)|$ =MAX $|\mu_{\underset{\sim}{A}}(x_i), \mu_{\underset{\sim}{B}}(x_i)|$ -MIN $|\mu_{\underset{\sim}{A}}(x_i), \mu_{\underset{\sim}{B}}(x_i)|$

(5.84) $\quad 0<e(\underset{\sim}{A} , \underset{\sim}{B})<1.$

$e^2(\underset{\sim}{A} , \underset{\sim}{B})$ is called the relative euclidean norm:

(5.85) $e^{2}(\underset{\sim}{A},\underset{\sim}{B}) = \frac{e\,2(\underset{\sim}{A},\underset{\sim}{B})}{n} = \frac{1}{n}\sum_{i=1}^{n}(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i))^2$

One will not be astonished that, in the particular case where $\mu_{\underset{\sim}{A}}(x_i), \mu_{\underset{\sim}{B}}(x_i) \in [0 , 1]$

(5.86) $e^{2}(\underset{\sim}{A},\underset{\sim}{B}) = d(\underset{\sim}{A},\underset{\sim}{B})$.

(5.87) $e^{2}(\underset{\sim}{A},\underset{\sim}{B}) = \delta(\underset{\sim}{A},\underset{\sim}{B})$.

These correspond to the boolean property

(5.88) $a^{2} = a.\quad a \in (0,1)$

Thus one may say that (5.76) and (5.81) generalize the notions of Hamming distance, absolute or relative, (5.47), and (5.53); one may not classify the euclidean norm as a distance since this norm does not satisfy the inequality (5.51) of the notion of distance.

The choice of a notion of distance, whether generalized (absolute or relative) Hamming or euclidean (absolute or relative), depends on the nature of the problem to be treated. These possess, respectively, advantages and inconveniences, which become evident in applications; we shall occupy ourselves with this in Volume III. One may, obviously, imagine and define other notions of distance,

Example. Let

(5.89) $\underset{\sim}{A}$ =

x_1	x_2	x_3	x_4	x_5	x_6	x_7
0,7	0,2	0	0,6	0,5	1	0

and

(5.90) $\underset{\sim}{B}$ =

x_1	x_2	x_3	x_4	x_5	x_6	x_7
0,2	0	0	0,6	0,8	0,4	1

.

One has

$d(\underset{\sim}{A},\underset{\sim}{B}) = |0.7 - 0.2| + |0.2 - 0| + |0 - 0| + |0.6 - 0.6| + |0.5 - 0.8| + |1 - 0.4| + |0 - 1|$

(5.91) $= 0.5 + 0.2 + 0 + 0 + 0.3 + 0.6 + 1 = 2.6$

(5.92) $\delta(\underset{\sim}{A},\underset{\sim}{B}) = \frac{1}{7}d(\underset{\sim}{A},\underset{\sim}{B}) = \frac{2.6}{7} = 0.37$.

(5.93) $e^{2}(\underset{\sim}{A},\underset{\sim}{B}) = (0.7 - 0.2^2 + (0.2 - 0)^2 + (0 - 0)^2 + (0.6 - 0.6)^2 + (0.5 - 0.8)^2 + (1 - 0.4)^2 + (0 - 1)^2$

$=(0.5)^2+(0.2)^2+(0)^2+(0)^2+(0.3)^2 + (0.6)^2 + (1)^2$

$=1.74.$

(5.94) $e(\underset{\sim}{A},\underset{\sim}{B}) = \sqrt{1.74} = 1.32$

(5.95) $e(\underset{\sim}{A},\underset{\sim}{B}) = \frac{e(\underset{\sim}{A},\underset{\sim}{B})}{\sqrt{n}} = \frac{1.32}{\sqrt{7}} = 0.49$

Case of a nonfinite reference set. The distances d($\underset{\sim}{A}$,$\underset{\sim}{B}$) and e($\underset{\sim}{A}$,$\underset{\sim}{B}$), and thus evidently the norm e ^{2}($\underset{\sim}{A}$,$\underset{\sim}{B}$), may be extended to the case where the reference set is not finite (denumerable or not), with the reservation, of course, that the corresponding summations be convergent.

If E is denumerable, one writes

(5.96) $$d(\underset{\sim}{A},\underset{\sim}{B}) = \sum_{i=1}^{n} |\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i)|$$

if this series is convergent.

If E = R, one writes

(5.97) $$d(\underset{\sim}{A},\underset{\sim}{B}) = \int_{-\infty}^{+\infty} |\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i)| \, dx$$

if this integral is convergent.

And similarly (see Figure 5.6),

(5.98) $$e(\underset{\sim}{A},\underset{\sim}{B}) = \sqrt{\sum_{i=1}^{n}(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i))^2}$$

if this series is convergent.

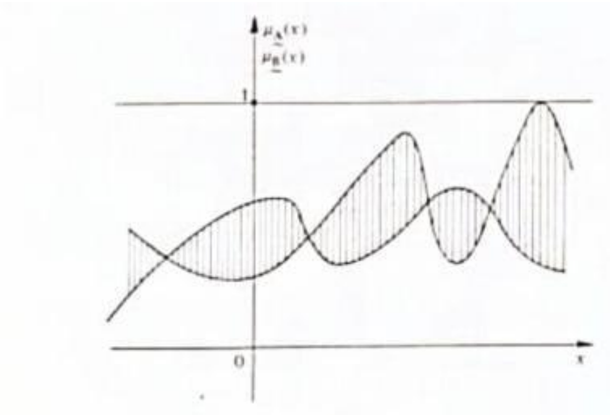


Fig. 5.6

And

(5.99) $$e(\underset{\sim}{A},\underset{\sim}{B}) = \sqrt{\int_{-\infty}^{+\infty}(\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underset{\sim}{B}}(x_i))^2 dx}$$

if this integral is convergent

Generally, δ($\underset{\sim}{A}$,$\underset{\sim}{B}$) and e($\underset{\sim}{A}$,$\underset{\sim}{B}$) are not used in the case of a nonfinite reference set, but one may, if necessary, at the cost of using a different definition or interposing other notions of convergence.

If one considers the case where E $\subset$ R is bounded above and below, then the integral (5.97) is convergent and likewise (5.98); then d($\underset{\sim}{A}$,$\underset{\sim}{B}$) and e($\underset{\sim}{A}$,$\underset{\sim}{B}$) are always finite.

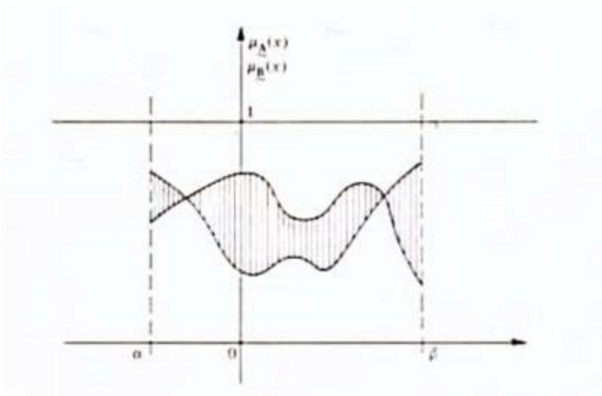


Fig. 5.7

In this case one will always be able to define $\delta(\underset{\sim}{A}, \underset{\sim}{B})$ and e($\underset{\sim}{A}, \underset{\sim}{B}$) (Figure 5.7):

(5.100) $$\delta(\underset{\sim}{A}, \underset{\sim}{B}) = \frac{d(\underset{\sim}{A}, \underset{\sim}{B})}{\beta-\alpha}$$

(5.101) $$e(\underset{\sim}{A}, \underset{\sim}{B}) = \frac{e(\underset{\sim}{A}, \underset{\sim}{B})}{\sqrt{\beta-\alpha}}$$

Ordinary subset nearest to a fuzzy subset. We pose the following question: Which is the ordinary subset (or subsets) A that has, with respect to a given fuzzy subset $\underset{\sim}{A}$, the smallest euclidean distance (or, if one wishes, the smallest norm)?

It is trivial to prove that this will be the ordinary subset, denoted A, such that

(5.102)
$$\begin{aligned} \mu_{\underset{\sim}{A}}(x_i) &= 0 \quad \text{if } \mu_{\underset{\sim}{A}}(x_i) < 0.5 \\ &= 1 \quad \text{if } \mu_{\underset{\sim}{A}}(x_i) > 0.5 \\ &= 0 \text{ or } 1 \text{ if } \mu_{\underset{\sim}{A}}(x_i) = 0.5 \end{aligned}$$

Where, by convention, we take $\mu_{\underset{\sim}{A}}(x_i) = 0$ if $\mu_{\underset{\sim}{A}}(x_i) = 0.5$.

Example. Let

(5.103) $\underset{\sim}{A}$ =

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0.2	0.8	0.5	0.3	1	0	0.9	0.4

Then one has

(5.104) $\underset{\approx}{A}$ =

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0	1	0	0	1	0	1	0

Index of fuzziness. One may consider, among others, two indexes of fuzziness: the linear index of fuzziness, defined with respect to the generalized relative Hamming distance, and the quadratic index of fuzziness, defined with respect to the relative euclidean distance. One designates these respectively by $\nu(\underset{\sim}{A})$ and $\eta(\underset{\sim}{A})$

(5.105) $\nu(\underset{\sim}{A}) = \frac{4}{n}\ d(\underset{\sim}{A}, \underset{\sim}{A})$

(5.106) $\eta(\underset{\sim}{A}) = \frac{2}{\sqrt{n}} c(\underset{\sim}{A}, \underset{\sim}{A})$

The number 2 appears in the numerator in order to obtain

(5.107) $0 < \nu(\underset{\sim}{A}) < 1$ and

(5.108) $0 < \eta(\underset{\sim}{A}) < 1$

because

(5.109) $0 < \delta\, (\underset{\sim}{A}, \underset{\sim}{A}) < \frac{1}{2}$ and

(5.110) $0 < c\, (\underset{\sim}{A}, \underset{\sim}{A}) < \frac{1}{2}$

The notion of the subset closest to a given fuzzy subset and the notion of the index of fuzziness may be extended to the case of a nonfinite reference set, with reservations-for example, concerning the index of fuzziness, that the summation be convergent. We shall consider the case of reference set E = [a,b]∈ R.

Figure 5.8 indicates how to evaluate the ordinary subset nearest and, from there, the index of fuzziness. For example, formula (5.105) gives

(5.111) $\nu(\underset{\sim}{A}) = \frac{2}{b-a} \int_a^b |\psi_{\underset{\sim}{A}}(x) - \mu_{\underset{\sim}{A}}(x)|\, dx$

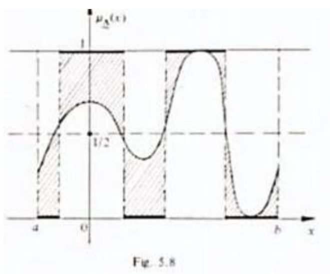


Fig. 5.8

Principal properties concerning the nearest ordinary subset. The following proper- ties may be easily verified:

(5.112) $\underline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \underline{\underset{\sim}{A}} \cap \underline{\underset{\sim}{B}}$

(5.113) $\underline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \underline{\underset{\sim}{A}} \cup \underline{\underset{\sim}{B}}$

Another interesting property is

(5.115) $\forall\, x_i \in E : |\mu_{\underset{\sim}{A}}(x_i) - \mu_{\underline{\underset{\sim}{A}}}(x_i)| = \mu_{\underset{\sim}{A} \cap \underline{\underset{\sim}{A}}}(x_i)$.

which is proved with reference to properties (5.112) and (5.114) We shall see an example by reconsidering (5.103) and (5.104):

(5.116) $\underline{\underset{\sim}{A}} =$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0,8	0,2	0,5	0,7	0	1	0,1	0,6

(5.117) $\underset{\sim}{A} \cap \underline{\underset{\sim}{A}} =$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0,2	0,2	0,5	0,3	0	0	0,1	0,4

One sometimes calls the fuzzy subset whose membership function is $2\mu_{\underset{\sim}{A} \cap \underline{\underset{\sim}{A}}}(x)$ the vectorial indicator of fuzziness. Thus for (5.103) one has

Equation (5.114) has been omitted.

Proposed by M. Nadler, research engineer at Honeywell Bull Cie.

(5.118)

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
0,4	0,4	1	0,6	0	0	0,2	0,8

Formula (5.105) may be written more conveniently as

(5.119) $P(\underset{\sim}{A}) = \frac{2}{n}\sum_{i=1}^{n} \mu_{\underset{\sim}{A} \cap \underline{\underset{\sim}{A}}}(x_i)$

One again has

(5.120) $P(\underline{\underset{\sim}{A}}) = P(\underset{\sim}{A})$

One may ask the following interesting question: Suppose $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are two fuzzy subsets of the same reference set E; then do $\underset{\sim}{A} \cap \underset{\sim}{B}$ or $\underset{\sim}{A} \cup \underset{\sim}{B}$ have indexes of fuzziness larger (or smaller) than $\underset{\sim}{A}$ or/and $\underset{\sim}{B}$? The following counterexamples show that, unfortunately, one may not say anything on this subject:

(5.121)

$\underset{\sim}{A} =$

x_1	x_2	x_3
0,2	0,6	0,1

$\nu(\underset{\sim}{A}) = 0{,}46$

$\underset{\sim}{B} =$

x_1	x_2	x_3
0,6	0,3	0,8

$\nu(\underset{\sim}{B}) = 0{,}60$

$\underset{\sim}{A} \cap \underset{\sim}{B} =$

x_1	x_2	x_3
0,2	0,3	0,1

$\nu(\underset{\sim}{A} \cap \underset{\sim}{B}) = 0{,}40$

(5.122)

$\underset{\sim}{A}' =$

x_1	x_2	x_3
0,8	0,6	0,8

$\nu(\underset{\sim}{A}') = 0{,}53$

$\underset{\sim}{B}' =$

x_1	x_2	x_3
0,4	0,7	0,2

$\nu(\underset{\sim}{B}') = 0{,}60$

$\underset{\sim}{A}' \cap \underset{\sim}{B}' =$

x_1	x_2	x_3
0,4	0,6	0,2

$\nu(\underset{\sim}{A}' \cap \underset{\sim}{B}') = 0{,}66$

The same holds concerning $\underset{\sim}{A} \cup \underset{\sim}{B}$, also unfortunately, and similarly also using η(A).

Since we have seen that a fuzzy subset and its complement have the same index of fuzziness, one then sees that each operation (∩, ∪,) does not ensure any systematic effect of increasing or decreasing fuzziness.

Evaluation of fuzziness through entropy. We here restrict ourselves to the case of a finite reference set. We know that the entropy of a system measures the degree of dis- order of the components of the system with respect to the probabilities of state.

Consider N states ϵ_1 ,ϵ_2 ,... ϵ_N of a system with which are associated the probabilities $P_1, P_2, \dots, P_N$, then the entropy of the system is defined by

Taken to a multiplicative coefficient K, to be placed before the summation sign ∑.In indicates the naperian logarithm, using the base e.

(5.123) $|(P_1, P_2, \dots, P_N)| = -\sum_{i=1}^{n} P_i \ln P_i$

It is easy to show that

(5.124) H = 0 (H minimal) for $P_r = 1$. r ϵ (1 , 2, … , N)

$P_r = 0$ I ≠ r.

(5.125) H= In N (H maximal) for $P_1 = P_2 = \dots = P_N = P = \frac{1}{N}$

If we take the formula

(5.126) $H(P_1, P_2, \dots, P_N) = \frac{1}{\ln N} \sum_{i=1}^{n} P_i \ln P_i$

the entropy is then a quantity that varies between 0 and 1:

(5.127) $H_{min} = 0$ and $H_{max} = 1$.

We shall see how to use this notion to evaluate the fuzziness of a subset. Consider a fuzzy subset $\underset{\sim}{A}$

(5.128) $\mu_{\underset{\sim}{A}}(x_1) = 0.7$, $\mu_{\underset{\sim}{A}}(x_2) = 0.9$,$\mu_{\underset{\sim}{A}}(x_3) = 0$, $\mu_{\underset{\sim}{A}}(x_4) = 0.6$, $\mu_{\underset{\sim}{A}}(x_5) = 0.5$, $\mu_{\underset{\sim}{A}}(x_6) = 0.1$

Putting

(5.129) $\pi_{\underset{\sim}{A}}(x_i) = \frac{\mu_{\underset{\sim}{A}}(x_i)}{\sum_{i=1}^{n} \mu_{\underset{\sim}{A}}(x_i)}$

one obtains

(5.130) $\pi_{\underset{\sim}{A}}(x_1) = \frac{7}{37}, \ \pi_{\underset{\sim}{A}}(x_2) = \frac{9}{37}\ , \ \pi_{\underset{\sim}{A}}(x_3) = 0\ , \ \pi_{\underset{\sim}{A}}(x_4) = \frac{6}{37}\ , \ \pi_{\underset{\sim}{A}}(x_5) = \frac{5}{37}\ ,$ $\pi_{\underset{\sim}{A}}(x_6) = \frac{10}{37}$

Then

(5.131) $H|(\pi_1, \pi_2, \pi_3, \dots, \pi_n)| = - \frac{1}{\ln 6} \sum_{i=1}^{n} \pi_{\underset{\sim}{A}}(x_i) \ln \pi_{\underset{\sim}{A}}(x_i)$

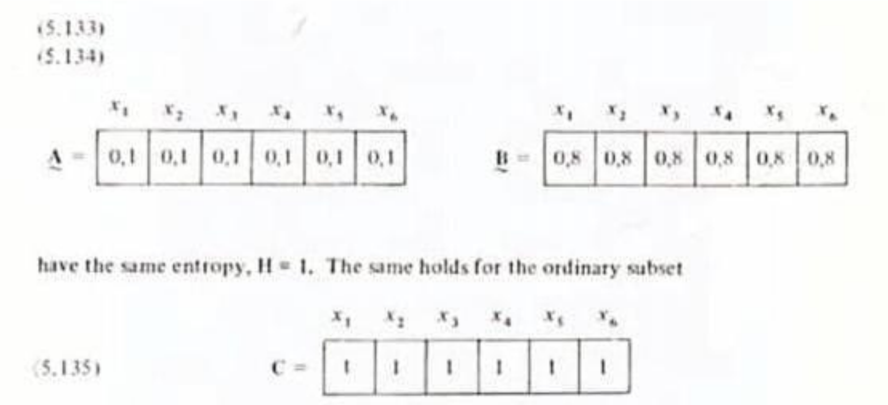
$= - \frac{1}{\ln 6} (\frac{7}{37} \ln \frac{7}{37} + \frac{9}{37} \ln \frac{9}{37} + \frac{6}{37} \ln \frac{6}{37} + \frac{5}{37} \ln \frac{5}{37} + \frac{10}{37} \ln \frac{10}{37} = 0.89$

The general formula permitting the calculation of the entropy from the fuzziness may be rewritten as

(5.132) $H(\pi_{\underset{\sim}{A}}(x_1), \pi_{\underset{\sim}{A}}(x_2), \dots \pi_{\underset{\sim}{A}}(x_n)) = - \frac{1}{\ln 6} \sum_{i=1}^{n} \pi_{\underset{\sim}{A}}(x_i)) \ln \pi_{\underset{\sim}{A}}(x_i)$

$= \frac{1}{\ln N \sum_{i=1}^{N} \mu_{\underset{\sim}{A}}(x_i)} [(\sum_{i=1}^{N} \mu_{\underset{\sim}{A}}(x_i) . (\ln \sum_{i=1}^{N} \mu_{\underset{\sim}{A}}(x_i)) - \sum_{i=1}^{N} \mu_{\underset{\sim}{A}}(x_i)) \ln \mu_{\underset{\sim}{A}}(x_i)$

We remark that this method of calculating fuzziness through entropy does not depend on accounting the effective values of μ but their relative values. Thus, the two fuzzy subsets below


(5.133)
(5.134)

$\underset{\sim}{A}$ =

x_1	x_2	x_3	x_4	x_5	x_6
0,1	0,1	0,1	0,1	0,1	0,1

$\underset{\sim}{B}$ =

x_1	x_2	x_3	x_4	x_5	x_6
0,8	0,8	0,8	0,8	0,8	0,8

have the same entropy, H = 1. The same holds for the ordinary subset

(5.135) C =

x_1	x_2	x_3	x_4	x_5	x_6
1	1	1	1	1	1

All ordinary subsets having a single nonzero element have entropy 0. Finally, the empty subset will always have an entropy equal to 1.

Entropy may be used in the theory of fuzzy subsets, but it is not a good indicatort: it relates to the theory of probabilities in systems, a different theory, as we will see in Section 40, than that we shall examine here. Sometimes it is possible to have some rapprochement, but only sometimes.

Ordinary subset of level a. Let a ∈ [0, 1]; one will call the ordinary subset of level a of a fuzzy subset $\underset{\sim}{A}$, the ordinary subset

(5.136) $$A_\alpha = \{x \mid \mu_{\underset{\sim}{A}}(x) > \alpha\}.$$

Example 1. Let

(5.137)
$$\underset{\sim}{A} = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline \multicolumn{1}{|c|}{0{,}8} & \multicolumn{1}{c|}{0{,}1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0{,}3} & \multicolumn{1}{c|}{0{,}6} & \multicolumn{1}{c|}{0{,}2} & \multicolumn{1}{c|}{0{,}5} \\ \hline \end{array} .$$

Since publication of the present work (first French edition), A. De Luca and S. Termini [D2] have defined a new and interesting extension of the concept of entropy for fuzzy subsets,

One has

(5.138)
$$A_{0,3} = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \end{array} .$$

(5.139)
$$A_{0,55} = \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{0} \\ \hline \end{array} .$$

Example 2. In Figure 5.9 we present an example where the reference set is R^*.

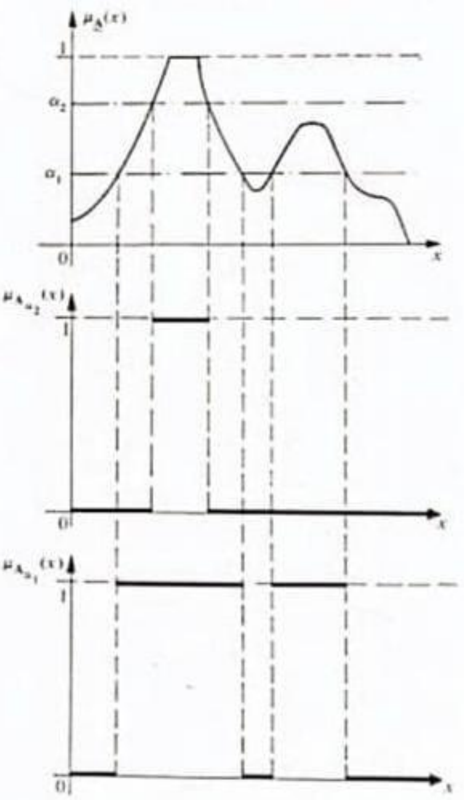


Fig. 5.9

Important Property: At once we have the evident property

$\alpha_2 > \alpha_1 = A_{\alpha_2} \subset A_{\alpha_1}$.

We now consider an important theorem:

Decomposition theorem. Any fuzzy subset $\underset{\sim}{A}$ may be decomposed in the following form, clearly as products of ordinary subsets by the coefficients a_1:

(5.141) $\underset{\sim}{A}$ =MAX $(\alpha_1 A_{a_1}, \alpha_2 A_{a_2}, \ldots, \alpha_n A_{a_n})$, $0 < a_1 < 1$, $i = 1, 2, \ldots, n$.

The proof is immediate:

(5.142) $\mu_{A_{a_i}}(x) = 1$ if $\mu_{\underset{\sim}{A}}(x) > a_i$.

$= 0$ if $\mu_{\underset{\sim}{A}}(x) < a_i$.

Thus, the membership function of A may be written

(3.143) $\mu(x) = \underset{a_i}{MAX} [\alpha_i A_{a_i}]$

$= \underset{a_i < \mu_{\underset{\sim}{A}}(x)}{MAX} [\alpha_i]$

$= \mu_{\underset{\sim}{A}}(x)$

Example I

(5,144)

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 0{,}2 & 0 & 0{,}5 & 1 & 0{,}7 \end{array} = \mathrm{MAX}\Big((0{,}2)\,.\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 1 & 0 & 1 & 1 & 1 \end{array},$$

$$(0{,}5)\,.\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 0 & 0 & 1 & 1 & 1 \end{array}, \quad (0{,}7)\,.\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 0 & 0 & 0 & 1 & 1 \end{array},$$

$$(1)\,.\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline 0 & 0 & 0 & 1 & 0 \end{array}\Big).$$

Example 2. The decomposition formula (5.142) is still valid when the reference set has the power of the continuum. Let, for example,

(5.145) $\mu_{\underset{\sim}{A}}(x) = 1 - \frac{1}{1+x^2}$, $x \in R^*$.

Considering the interval [a, 1], where 0 < a < 1, we may write

(5.146) $\mu_{A_\alpha}(x)$ = 1 if $\mu_{\underset{\sim}{A}}$ (x)∈[0 , 1]

= 0 if $\mu_{\underset{\sim}{A}}$ (x)∈ [0 , 1]

Thus, in the given example

(5.147) $\mu_{A_\alpha}(x)$ = 1 if $x > \sqrt{\frac{\alpha}{1-\alpha}}$

= o if $x < \sqrt{\frac{\alpha}{1-\alpha}}$

And for all arbitrary sets of value a, 0 < a<1, one may decompose (5.145).

Synthesis of a fuzzy subset by joining ordinary subsets. The decomposition theorem may be applied not only for analysis but also for synthesis: If one then considers a sequence of ordinary subsets

(5.148) $A_1 \subset\subset A_2 \subset\subset \dots \subset\subset A_n$

and attributes α_1, to A_1 , α_2, to A_2 , …. , α_n, to A_n , with

(5.149) $\alpha_1 > \alpha_2 > \dots > \alpha_n$.

then one obtains a fuzzy subset with the aid of (5.140).

6. SET OF FUZZY SUBSETS FOR E AND M FINITE

We restrict ourselves to the case where E and M are finite. Recall the definition of the set of subsets (or power set) of a set by considering a simple example. Let

(6.1) $E = \{x_1 , x_2 , x_3\}$

Then

(6.2) $P(E) = \{\varphi , \{ x_1\} , \{ x_2\} , \{ x_3\} , \{ x_1 , x_2\} , \{ x_1 , x_3\} , \{ x_2 , x_3\} , E \}$.

There are $2^3 = 8$ elements in this set. More generally, for a set

(6,3) $E = \{ x_1 , x_2 , \dots , x_n\}$

one may define 2^n elements in the same manner.

For fuzzy subsets, the power set or "set of fuzzy subsets" is presented in a different manner. First we consider an example. Let

(6,4) $E = \{ x_1 , x_2\}$

(6,5) and $M = \{0 , \frac{1}{2} , 1 \}$

The set of fuzzy subset $\underset{\sim}{R}$ (E) will be

(6.6) $\underset{\sim}{R}$ (E) = {{(x_1|0) , (x_2|0)} , {(x_1|0) , (x_2|0.5)} , {(x_1|0.5) , (x_2|0)}, {{(x_1|0.5) , (x_2|0.5)}, {(x_1|0) , (x_2|1)}, {(x_1|1) , (x_2|0)}, {(x_1|1) , (x_2|0.5)}, {{(x_1|0.5) , (x_2|1)}, {(x_1|1) , (x_2|1)}.

More generally ,if

(6.6a) card E = n and card M = m

Where card means cardinality of that is gives the number of elements of the set, then

(6.7) card $\underset{\sim}{R}$ (E) = m^n.

It follows that card $\underset{\sim}{R}$ (E) is finite if and only if m and n are finite. The set R(E) contains 2^n ordinary subsets,

Consider another example for better comparison with (6.2):

(6.8) E = {x_1,x_2,x_3} and M = {0,$\frac{1}{2}$,1}.

(6.9) $\underset{\sim}{R}$ (E) = {{(x_1|0) , (x_2|0), (x_3|0)} , {(x_1|0) , (x_2|0), (x_3|$\frac{1}{2}$} , {(x_1|0) , (x_2|$\frac{1}{2}$),(x_3|0)}, {{(x_1|0.5) , (x_2|0),(x_3|0)}, {(x_1|0) , (x_2|0),(x_3|1)}, {(x_1|0) , (x_2|0.5),(x_3|0.5)}, {(x_1|0) , (x_2|1),(x_3|0)},………. {(x_1|1) , (x_2|0.5),(x_3|1)}, {(x_1|1) , (x_2|1),(x_3|0.5), {(x_1|1),(x_2|1),(x_3|1)}.

It is well known that the structure of a power set $\underset{\sim}{R}$ (E) of a set is a distributive and complementary lattice that is a Boolean lattice. The set of fuzzy subsets $\underset{\sim}{R}$ (E) however has the structure of a vectorial lattice that is distributive but not complementary.

Note that one always has

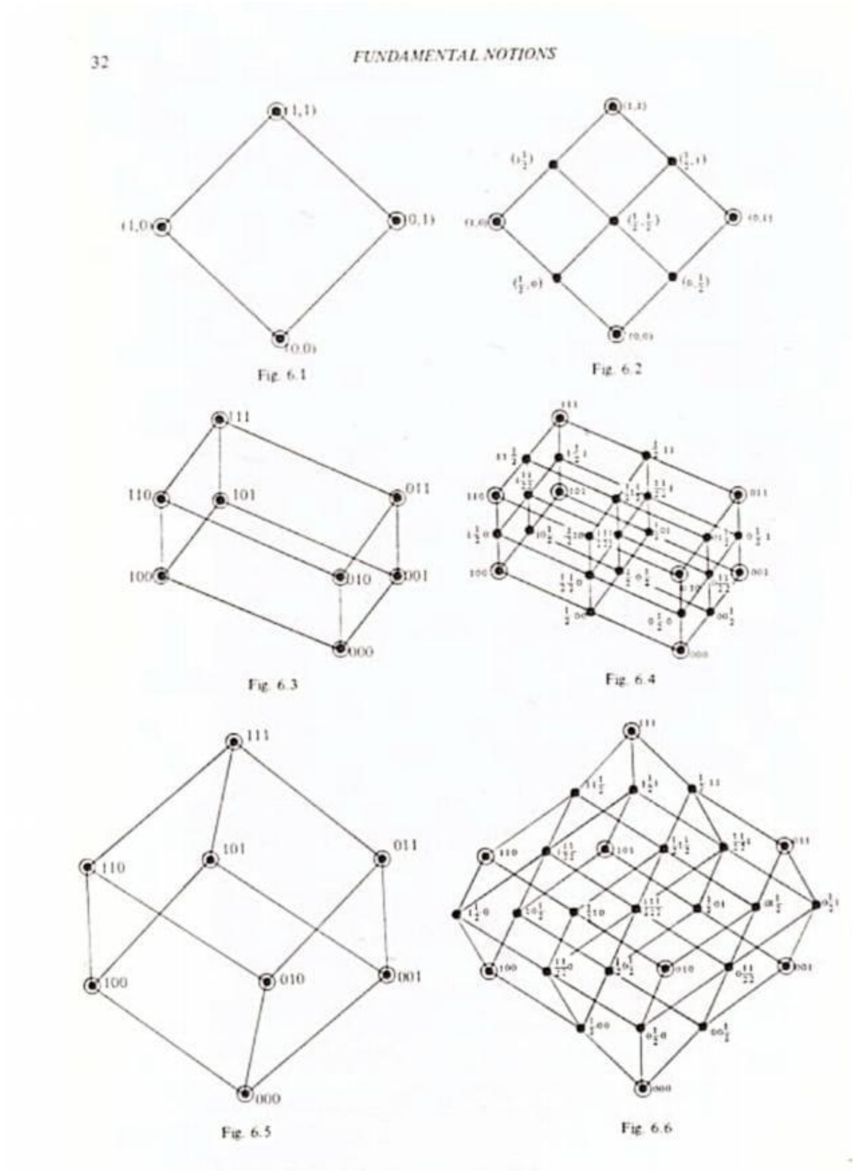
R(E) ⊂ $\underset{\sim}{R}$ (E)

Thus in considering (6.4) and (6.5) , one may write

R(E)=$\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\} =$

$\{\{(x_1|0), (x_2|0)\}, \{(x_1|1), (x_2|0)\}, \{(x_1|0), (x_2|1)\}, \{(x_1|1), (x_2|1)\}\}$

This is ofcourse a subset of (6,6), One will see this more clearly later in figure 6.1 and 6.2,6.3 and 6.4 and 6.5 and 6.6.



PROPERTIES OF THE SET OF FUZZY SUBSETS

Recall that in a distributive lattice if the complement of an element exists, it is unique; and that this is the case for a vectorial lattice. The complementation consider here has a different sense from the given in (5,17).

The complementation does not necessarily give, as one says complement in a lattice, $A \cap \bar{A} = E$ and $A \cup \bar{A}$ = E this is all the difference ,but it is primary.

In figures 6.1-6.6 we present several simple examples where , in order to simplify the labels , The fuzzy subsets are represented by their respective membership function.

Figure 6.2 : E = $\{x_1, x_2\}$, M = {0,0.5,1}. The figure represent a vector lattice of fuzzy subsets, and figures 6.1 a Boolean lattice of ordinary sets.

Figure 6.4: E = $\{x_1, x_2, x_3\}$, M = {0,0.5,1}. The figure represent a vector lattice of fuzzy subsets, and figures 6.3 a Boolean lattice of ordinary sets

Figure 6.6 This is another representation of the vector lattice of figure 6.4, to the left of which has been placed a Boolean lattice of ordinary sets (figure 6.5)

7. PROPERTIES OF THE SET OF FUZZY SUBSETS

Recall that the principal properties of the power set of an ordinary set E are as follows. Given $A \subset E, B \subset E, C \subset E$, One has :

(7.1) $A \cap B = B \cap A$,

(7.2) $A \cup B = B \cup A$. Commutativity properties

(7.3) $(A \cap B) \cap C = A \cap (B \cap C)$,

(7.4) $(A \cup B) \cup C = A \cup (B \cup C)$. Associativity properties

(7.5) $A \cap A = A$,

(7.6) $A \cup A = A$. Idempotence

(7.7) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

(7.8) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Distributivity of intersection with respect to Union and union with respect to Intersection

(7.9) $A \cap \bar{A} = \emptyset$

(7.10) $A \cup \bar{A} = E$

(7.11) $A \cap \emptyset = \emptyset$

(7.12) $A \cup \emptyset = A$

(7.13) $A \cap E = A$

(7.14) $A \cup E = E$

(7.15) $\bar{\bar{A}} = A$ Involution

(7.16) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

(7.17) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ De Morgan's theorems

If $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ are fuzzy subsets of E, all the properties (7.1 – 7.17) are satisfied except (7.9) and (7.10). One may define a unique complement, but the properties (7.9) an (7.10) hold only for ordinary subsets.

(7.18)	$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$	Commuatativity
(7.19)	$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}$	
(7.20)	$(\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C} = \underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C})$	Associativity
(7.21)	$(\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C} = \underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C})$	
(7.22)	$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$	Idempotence
(7.23)	$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A}$	
(7.24)	$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C})$	Distributivity
(7.25)	$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$	
(7.26)	$\underset{\sim}{A} \cap \emptyset = \emptyset$	Where $\emptyset$ is the ordinary set such that $\forall x_i \in E$: μ_R(xi)=0
(7.27)	$\underset{\sim}{A} \cup \emptyset = \underset{\sim}{A}$	
(7.28)	$\underset{\sim}{A} \cap E = \underset{\sim}{A}$	where E is the ordinary set such that $\forall x_i \in E$: μ_R(xi)=1
(7.29)	$\underset{\sim}{A} \cup E = E$	
(7.30)	$\overline{(\overline{\underset{\sim}{A}}))} = \underset{\sim}{A}$	Involution
(7.31)	$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}}$	De-Morgan's theorems for the case of fuzzy subsets
(7.32)	$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}}$	

Thus, we stress: All the properties of an ordinary power set are found again in a power set of fuzzy subsets (except (7.9) and (7.10). Thus, we no longer have an algebra in the sense of the theory of ordinary sets; the structure is that of a vector lattice.

8. ALGEBRAIC PRODUCT AND SUM OF TWO FUZZY SUBSETS

Let E be a set and $M = [0,1]$ its associated membership set, let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two fuzzy subsets of E; one defines the algebraic product of $\underset{\sim}{A}$ and $\underset{\sim}{B}$, denoted

(8.1) $$\underset{\sim}{A} \cdot \underset{\sim}{B}$$

in the following manner:

(8.2) $$\forall x \in E: \ \mu_{\underset{\sim}{A} \cdot \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x).$$

Likewise one defines the algebraic sum of these two subsets, denoted

(8.3) $$\underset{\sim}{A} \mp \underset{\sim}{B},$$

in the following manner:

(8.4) $$\forall x \in E: \ \mu_{\underset{\sim}{A} \mp \underset{\sim}{B}}(\mathrm{x}) = \mu_{\underset{\sim}{A}}(x) + \mu_{\underset{\sim}{B}}(x) - \mu_{\underset{\sim}{A}}(x) \cdot \mu_{\underset{\sim}{B}}(x).$$

Consider again the example in (5.25) – (5.27)

(8.5) $$\underset{\sim}{A} = \{(x_1|0.2), (x_2|0.7), (x_3|1), (x_4|0), (x_5|0.5)\}.$$

(8.6) $$\underset{\sim}{B} = \{(x_1|0.5), (x_2|0.3), (x_3|1), (x_4|0.1), (x_5|0.5)\}.$$

(8.7) $$\underset{\sim}{A} \cdot \underset{\sim}{B} = \{(x_1|0.10), (x_2|0.21), (x_3|1), (x_4|0), (x_5|0.25)\}.$$

(8.8) $$\underset{\sim}{A} \mp \underset{\sim}{B} = \{(x_1|0.60), (x_2|0.79), (x_3|1), (x_4|0.1), (x_5|0.75)\}.$$

We now make the following important remark:

If $M = \{0,1\}$, that is, if we are in the case of ordinary subsets, then

(8.9) $$A \cap B = A \cdot B,$$

(8.10) $$A \cup B = A \mp B.$$

In fact, if $\mu_A(x) \in \{0,1\}$ and $\mu_B(x) \in \{0,1\}$, the following tables are equivalent, but this is not true for $M \neq \{0, 1\}$, except in a few trivial cases.

(8.11)

MIN	0	1
0	0	0
1	0	1

is equivalent to

(.)	0	1
0	0	0
1	0	1

.

(8.12)

MAX	0	1
0	0	1
1	1	1

is equivalent to

(∔)	0	1
0	0	1
1	1	1

.

In the present work we use the algebraic product and sum operations rather in frequently , but these constitute an interesting direction for other research.

If one considers the two operations $\cdot$ and $\dotplus$on the power set of fuzzy subsets , only the following properties may be verified ; these are obviously more restricted than those for $\cap$ *and* $\cup$ for the power set of fuzzy subsets, and a fortiori those concerning $\cap$ *and* $\cup$ for ordinary power sets. One may easily verify

(8.13) $\underset{\sim}{A} \cdot \underset{\sim}{B} = \underset{\sim}{B} \cdot \underset{\sim}{A}$

(8.14) $\underset{\sim}{A} \dotplus \underset{\sim}{B} = \underset{\sim}{B} \dotplus \underset{\sim}{A}$ — Commuatativity

(8.15) $(\underset{\sim}{A} \cdot \underset{\sim}{B}) \cdot \underset{\sim}{C} = \underset{\sim}{A} \cdot (\underset{\sim}{B} \cdot \underset{\sim}{C})$

(8.16) $(\underset{\sim}{A} \dotplus \underset{\sim}{B}) \dotplus \underset{\sim}{C} = \underset{\sim}{A} \dotplus (\underset{\sim}{B} \dotplus \underset{\sim}{C})$ — Associativity

(8.17) $\underset{\sim}{A} \cdot \emptyset = \emptyset$ — Where $\emptyset$ is the ordinary set such that $\forall x_i \in E$: $\mu_R(x_i)=0$

(8.18) $\underset{\sim}{A} \dotplus \emptyset = \underset{\sim}{A}$

(8.19) $\underset{\sim}{A} \cdot E = \underset{\sim}{A}$ — where E is the ordinary set such that $\forall x_i \in E$: $\mu_R(x_i)=1$

(8.20) $\underset{\sim}{A} \dotplus E = E$

(8.21) $\overline{(\overline{\underset{\sim}{A}}))} = \underset{\sim}{A}$ — Involution

(8.22) $\overline{\underset{\sim}{A} \cdot \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cdot \overline{\underset{\sim}{B}}$

(8.23) $\overline{\underset{\sim}{A} \dotplus \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \dotplus \overline{\underset{\sim}{B}}$ — De-Morgan's theorems for the case of fuzzy subsets

Thus properties (7.5) and (7.6) (idempotence) are not satisfied, not are

(7.7) and (7.8) (distributivity), not likewise (7.9) and (7.10). This gives a noticeably poorer structure, especially because of the absence of distributivity. We shall show through several examples how to prove properties (8.13)-(8.23).

We prove (8.16), for example, by putting

(8.24) a = $\mu_A(x), b = \mu_B(x), c = \mu_C(x)$:

(8.25) $(A \dotplus B) \dotplus C = A \dotplus (B \dotplus C)$ is verified if

(8.26) $(a + b - ab) + c - (a + b - ab)c = a + (b + c + bc) - a(b + c - bc)$

is verified. By expanding the two members one has

(8.27)

$$a + b - ab + c - ac - bc + abc = a + b + c - bc - ab - ac + abc$$

The two sides are indeed identical. Thus, (8.25) is a correct formula.

We prove (8.22). The equation

(8.28) $\overline{\underset{\sim}{A} . B} = \overline{\underset{\sim}{A}} \dotplus \overline{B}$ is verified if

(8.29) 1-ab = (1-a) + (1-b) – (1-a) (1-b)

=1-a+1-b-1+a+b-ab

=1-ab.

We now prove that distributivity does not hold; for example,

(8.30) $\underset{\sim}{A} \cdot (\underset{\sim}{B} \dotplus \underset{\sim}{C}) - (\underset{\sim}{A} \cdot \underset{\sim}{B}) \pm (\underset{\sim}{A} \cdot \underset{\sim}{C})$

For the left-hand side of this equation , one must have

(8.31) $a.(b + c - bc) = ab + ac - abc$

(8.32) $ab + ac - (ab)\,(ac) \;=\; ab + ac - a2bc.$

These then prove nondistributivity.

We note that ∪ is not distributive with respect to . or $\dotplus$, and likewise ∩ ; but on the other hand one has

(8.33) $\underset{\sim}{A} \cdot (\underset{\sim}{B} \;\cup \underset{\sim}{C}) = (\underset{\sim}{A} \cdot \underset{\sim}{B}) \cup (\underset{\sim}{A} \cdot \underset{\sim}{C})$

(8.34)

$\underset{\sim}{A} \cdot (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cdot \underset{\sim}{B}) \cap (\underset{\sim}{A} \cdot \underset{\sim}{C})$

(8.35) $\underset{\sim}{A} \dotplus (\underset{\sim}{B} \;\cup \underset{\sim}{C}) = (\underset{\sim}{A} \dotplus \underset{\sim}{B}) \cup (\underset{\sim}{A} \dotplus \underset{\sim}{C})$

(8.36) $\underset{\sim}{A} \dotplus (\underset{\sim}{B} \;\cap \underset{\sim}{C}) = (\underset{\sim}{A} \dotplus \underset{\sim}{B}) \cap (\underset{\sim}{A} \dotplus \underset{\sim}{C})$

Index of fuzziness for a product.

It is possible to define an index of fuzziness for a product similar to (5.108) ; one puts

(8.37) $\eta(\underset{\sim}{A}) = \frac{4}{N}\sum_{i=1}^{N} \mu_{\underset{\sim}{A}\,.\underset{\sim}{\bar{A}}}\{x_i\}.$

Example : Let

(8.38)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\underset{\sim}{A} =$	0.7	0.2	0.9	1	0	0.4	1

(8.39)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\underset{\sim}{\bar{A}} =$	0.3	0.8	0.1	0	1	0.6	0

(8.40)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\underset{\sim}{\bar{A}}.\underset{\sim}{\bar{A}} =$	0.21	0.16	0.09	0	0	0.24	0

(8.41)

$$\eta(A) = \frac{4}{7}(0.21 + 0.16 + 0.09 + 0 + 0 + 0.24 + 0) = 0.40.$$

One may then, on the other hand, raise the following questions , as we have in section 5 for $\cap\, and\, \cup$: Are the indexes of fuzziness for $\underset{\sim}{A}\ .\underset{\sim}{B}$ or $\underset{\sim}{A} \mp \underset{\sim}{B}$ greater than or less than those of $\underset{\sim}{A}$ or/and $\underset{\sim}{B}$? Unfortunately, the operations . and $\dot{+}$ do not always modify the fuzziness in the same sense , as may be seen with examples.

General remark on the subject of fuzziness:

We have seen that each of the operations $\cup, \cap, ., \dot{+}$ does not systematically increase or decrease the fuzziness of a subset A in applying these operations with other subsets of the same reference set. It should be borne in mind that the membership function is supposed known in order to treat fuzzy subset adequately.

If * is one of the four operations considered above, one may not say, a priori, that for $\underset{\sim}{A} \subset E, \underset{\sim}{B} \subset E, \underset{\sim}{A}$ and $\underset{\sim}{B}$ arbitrary, whether $\upsilon(\underset{\sim}{A} * \underset{\sim}{B})$ is greater than or less $\upsilon(\underset{\sim}{A})$ or $\upsilon(\underset{\sim}{B})$.

One has the same situation in considering entropy it recurs their that if one wishes to increase or decrease the entropy II , it is necessary that one have knowledge of A: knowledge of II is not sufficient , one may surmise .

UNIT II

FUZZY GRAPHS

9. INTRODUCTION

The notions of graph, correspondence and relation play a fundamental role in applications of mathematics. The may be generalized with respect to the notion of fuzzy subsets. One will then discover some new and very interesting properties. For example, the notion of an equivalence class will be found to be replaced by that similitude, stronger and apt for representing some less precise but more often encountered situations. Preorder and order are likewise generalized; whereas some other relations such as resemblance and dissemblance, are defined. This, then is a new theory that may be formed with Fuzzy relations. It is only a beginning. It is likely that research on fuzzy concepts will develop progressively in importance and will permit at least good descriptions of complex phenomena, constrained until now to the specifications all or nothing.

10. FUZZY GRAPHS

Consider two sets E_1 and E_2 ; let x designate an element of E_1 and y an element of E_2. The set of ordered pairs (x,y) defines the product set $E_1 \times E_2$.

The Fuzzy subset $\underset{\sim}{G}$ such that

(11.1) For every $(x, y) \in E_1 \times E_2$: $\mu_{\underset{\sim}{G}}(x, y) \in M$

where M is the membership set of $E_1 \times E_2$, is called a fuzzy graph.

Example 1:

Let $E_1 = \{x_1, x_2, x_3\}$

and

$$E_2 = \{y_1, y_2\}$$

$$E_1 \times E_2 = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), \{(x_2, y_2), (x_3, y_1), (x_3, y_2)\}\ .$$

Set, in order to simplify notation,

$$\mu(x_i, y_j) = \mu_{\underset{\sim}{G}}(x_i, y_j)\ , i = 1, 2, 3, j = 1, 2.$$

which will be called the value of the ordered pair (x_i, y_j).

Consider for example :

$$\mu(x_1, y_1) = 0.3 \quad , \quad \mu(x_1, y_2) = 0.7 \quad , \mu(x_2, y_1) = 1$$
$$\mu(x_2, y_2) = 0 \quad , \quad \mu(x_3, y_1) = 0.5 \quad , \mu(x_3, y_2) = 0.2.$$

This function defines the fuzzy subset

$\underset{\sim}{G} = \{((x_1, y_1)|0.3),\ ((x_1, y_2)|0.7), ((x_2, y_1)|1), ((x_2, y_2)|0),$

$$((x_3, y_1)|0.5), ((x_3, y_2)|0.2)\}$$

This fuzzy subset may be represented by a matrix such as that shown in Figure 10.1.

	y_1	y_2
x_1	0.3	0.7
x_2	1	0
x_3	0.5	0.2

The graph

$$\underset{\sim}{G} \subset E_1 \times E_2$$

is a fuzzy graph.

The graph

$$\underset{\sim}{G} = \{((x_1, y_1)|0),\ ((x_1, y_2)|1), ((x_2, y_1)|1), ((x_2, y_2)|1),$$
$$((x_3, y_1)|1), ((x_3, y_2)|0)\}$$

is an ordinary graph in set theory (Figure 10.2)

	y_1	y_2
x_1	0	1
x_2	1	1
x_3	1	0

Figure 10.2

Example 2: Let $E_1 = E_2 = R^+$, where R^+ is the set of nonnegative real numbers. Let $x \in R^+$ and y$\in R^+$ and consider the product set $R^+ \times R^+$. Then the relation $y > x$ defines a fuzzy graph in ${R^+}^2$.

Suppose that one has a use for the function

$$\mu(x, y|y = x) = e^0 = 1$$
$$\mu(x, y|y = 2x) = e^{-1}.$$
$$\dots \dots \dots \dots \dots \dots \dots.$$
$$\mu(x, y|y = kx) = e^{-(k-1)},$$
$$\dots \dots \dots \dots \dots . k = 1,2,3,4, \dots \dots.$$

with $M = \{1, e^{-1}, e^{-2}, \dots \dots \dots \dots, e^{-(k-1)}, \dots \dots . .0\}$.

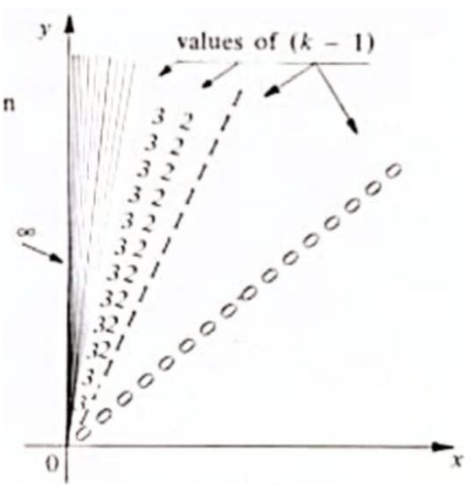


Figure 10.3

Figure 10.3 gives a visual representation of this fuzzy subset for the points $y = kx,\ k \geq 1$.

Example 3 (Berge graphs).

A graph in the sense of Berge is one such that

$$E_1 = E_2 = E$$

countable and is formed by the subset of ordered pairs

$$(x, y) \in G \subset E \times E,$$

such that

$$G \cap \bar{G} = \emptyset$$

and

$$G \cup \bar{G} = E \times E.$$

For such graphs, which evidently are only a particular case of the notion of graphs in set theory, one may define a generalization to fuzzy graphs. Thus, Figures 10.4, 10.6, 10.8 and 10.10 represent the same Berge fuzzy graphs, whereas Figures 10.5, 10.7, 10.9 and 10.11 show the same ordinary Berge graph.

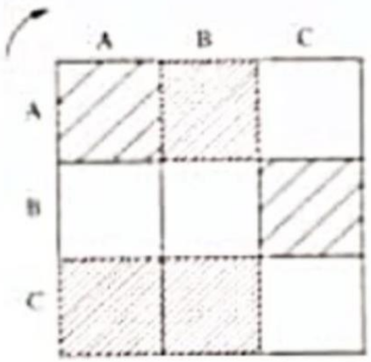


Figure 10.4

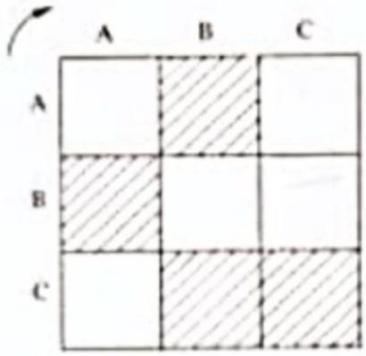


Figure 10.5

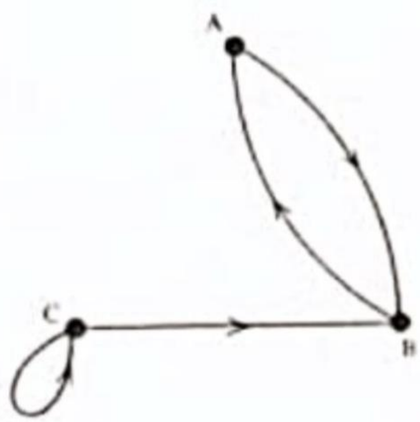


Figure 10.6

Figure 10.7

	A	B	C
A	0,5	1	0
B	0	0	0,5
C	1	1	0

Figure 10.8

	A	B	C
A	0	1	0
B	1	0	0
C	0	1	1

Figure 10.9

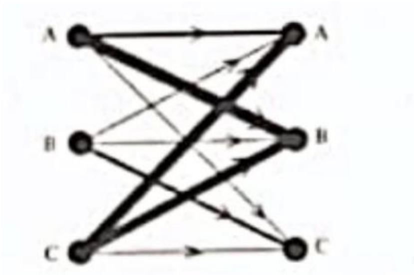


Figure 10.10

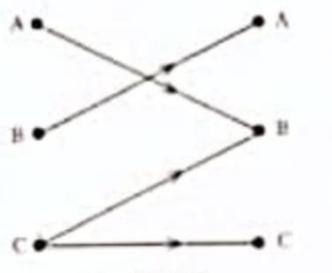


Figure 10.11

Using Berge's notation, for the ordinary graph in Figures 10.5, 10.7, 10.9 and 10.11, one puts

$$\Gamma\{A\} = \{B\},$$
$$\Gamma\{B\} = \{A\},$$
$$\Gamma\{C\} = \{B, C\}.$$

where $\Gamma\{X\}$ is called a multivalued mapping of {X} in its reference set E.

In the spirit of this notation, one will write for the fuzzy graph represented in various fashions in Figures 10.4, 10.6, 10.8 and 10.10.

$$\Gamma\{A\} = \{(A|0{,}5), (B|1), (C|0)\}$$
$$\Gamma\{B\} = \{(A|0), (B|0), (C|0{,}5)\}$$
$$\Gamma\{A\} = \{(A|1), (B|1), (C|0)\}$$

Example 4:

Figures 10.12 represent a fuzzy graph and Figure 10.13 an ordinary graph.

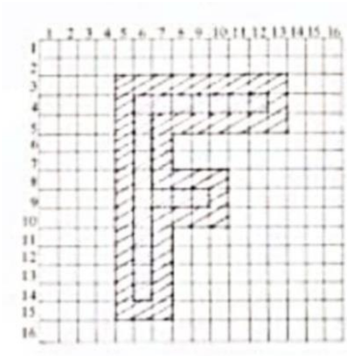


Figure 10.12

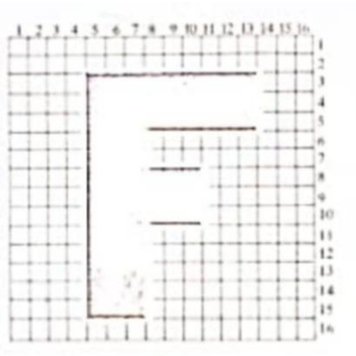


Figure 10.13

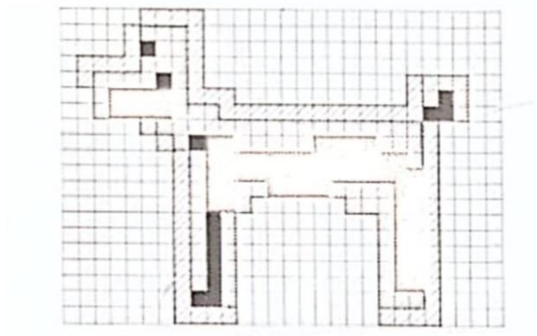


Figure 10.14

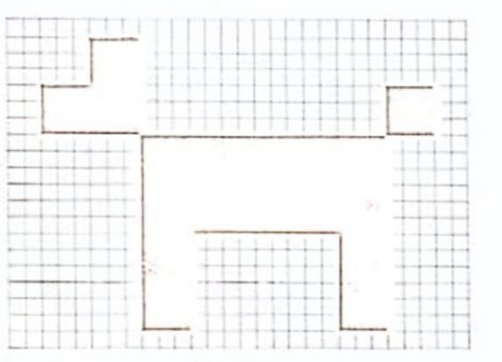


Figure 10.15

Also, Figure 10.14 represents a fuzzy graph and Figure 10.15 an ordinary graph.

Example 5:

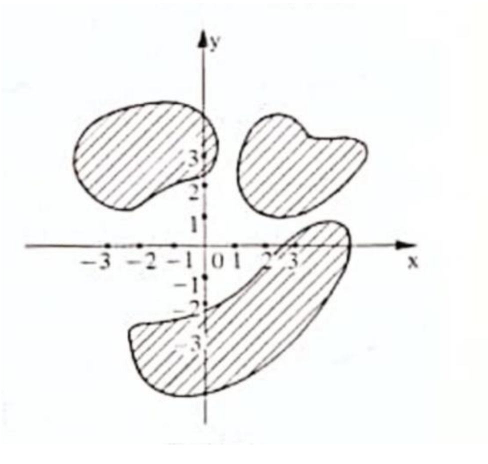


Figure 10.16

The shaded parts of Figure 10.16, where we attribute a value $\mu(x, y)$, to each point (x, y), represent a fuzzy graph.

Generalization :

What has been presented for a product set $E_1 \times E_2$ may be generalized for a product set

$$E_1 \times E_2 \times \dots \dots \dots \dots \dots \times E_n.$$

The fuzzy subset such that $x^{(i)} \in E_i, i = 1, 2, \dots \dots \dots, n,$

$$\forall \ \left(x^{(1)}, x^{(2)}, \dots \dots \dots \dots x^{(n)}\right) \in E_1 \times E_2 \times \dots \dots \dots \dots \dots \times E_n,$$

$$\mu\left(x^{(1)}, x^{(2)}, \dots \dots \dots \dots x^{(n)}\right) \in M,$$

where M is the membership set of $E_1 \times E_2 \times \dots \dots \dots \dots \dots \times E_n$, is called a fuzzy graph.

Example :

Let $E_1 = \{x_1, x_2\},\ E_2 = \{y_1, y_2\}\ ,\ E_3 = \{z_1, z_2\}, M = [0,1].$

$\underset{\sim}{G} = \{((x_1, y_1, z_1)|0.3)\ ,\ \left((x_1, y_1, z_2\ )\middle|0.2\right), \left((x_1, y_2, z_1)\middle|1\right), \left((x_1, y_2, z_2)\middle|0\right),$

$$\left((x_2, y_1, z_1)\middle|0\right), \left((x_2, y_1, z_2)\middle|0.1\right), \left((x_2, y_2, z_1)\middle|0.9\right), \left((x_2, y_2, z_2)\middle|0.7\right)\}$$

is a fuzzy graph in $E_1 \times E_2 \times E_3$.

11. FUZZY RELATION

As in done in the theory of ordinary sets, the notion of a fuzzy graph may be explained in terms of the notion of a fuzzy relation. Let P be a product set of n sets and M its membership function; a fuzzy relation is a fuzzy subset of P taking its values in M.

Example 1:

Let

$$E_1 = \{x_1, x_2, x_3\},$$

$$E_2 = \{y_1, y_2, y_3, y_4\},$$

$$M = [0,1].$$

	y_1	y_2	y_3	y_4	y_5
x_1	0	0	0.1	0.3	1
x_2	0	0.8	0	0	1
x_3	0.4	0.4	0.5	0	0.2

Figure 11.1

The table presented in 11.1 expresses a fuzzy 2-ary relation (which we may describe as binary if no confusion with other interpretations of the word binary is possible).

Example 2: Let

$$E_1 = E_2 = R$$

where $R = (-\infty, \infty)$, that is, the set of real numbers. Then the relation $y \lessdot x$, where $x \in R, y \in R,$ is a fuzzy relation in R^2.

For example, a subjective expression (that is, a valuation that may depend on a subjective estimate) of $y \lessdot x$ may be given by

$$\mu_{R^2}(x,y) = 0 \; if \; y \geq x,$$

$$= \frac{1}{1 + \frac{1}{(x-y)^2}} \; if \; y < x.$$

Notation :

A fuzzy relation in $E_1 \times E_2$ will be written as

$$x \in E_1 \,, \qquad y \in E_2 : x\underset{\sim}{R}y.$$

Symbols used for extrema :

In what follows we will use the symbols

$\bigvee_x$ *to represent the maximum with respect to an element or variable* x,

$\bigwedge_x$ *to represent the minimum with respect to an element or variable* x.

Thus, writing

$$\mu_1(x) = \bigvee_y \mu(x,y),$$

will be equivalent to

$$\mu_1(x) = \underset{y}{MAX}\, \mu(x,y).$$

Likewise, writing

$$\mu_2(x) = \bigwedge_{y} \mu(x,y),$$

will be equivalent to

$$\mu_2(x) = \underset{y}{MIN}\, \mu(x,y).$$

Projection of a fuzzy relation:

The membership function

$$\mu_{\underset{\sim}{R}}{}^{(1)}(x) = \bigvee_{y} \mu_{\underset{\sim}{R}}(x,y)$$

defines the first projection of $\underset{\sim}{R}$.

The second projection of the first project (or vice versa) will be called the global projection of the fuzzy relation and will be denoted by $h\left(\underset{\sim}{R}\right)$. Thus,

$$h\left(\underset{\sim}{R}\right) = \bigvee_{x} \bigvee_{y} \mu_{\underset{\sim}{R}}(x,y)$$
$$= \bigvee_{y} \bigvee_{x} \mu_{\underset{\sim}{R}}(x,y).$$

If $h\left(\underset{\sim}{R}\right) = 1$, the relation is said to be normal. If $h\left(\underset{\sim}{R}\right) < 1$, the relation is called subnormal.

Example 1:

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4	1st proj.
x_1	0,1	0,2	1	0,3	1
x_2	0,6	0,8	0	0,1	0,8
x_3	0	1	0,3	0,6	1
x_4	0,8	0,1	1	0	1
x_5	0,9	0,7	0	0,5	0,9
x_6	0,9	0	0,3	0,7	0,9
2nd proj.	0,9	1	1	0,7	1

Figure 11.2 global projection

We calculate the first projection:

$$\mu_{\underset{\sim}{R}}{}^{(1)}(x) = \bigvee_y \mu_{\underset{\sim}{R}}(x, y).$$

$$\mu_{\underset{\sim}{R}}{}^{(1)}(x_1) = \bigvee_y \mu_{\underset{\sim}{R}}(x_1, y) = MAX\{0.1, 0.2, 1, 0.3\} = 1.$$

$$\mu_{\underset{\sim}{R}}{}^{(1)}(x_2) = \bigvee_y \mu_{\underset{\sim}{R}}(x_2, y) = MAX\{0.6, 0.8, 0, 0.1\} = 0.8.$$

… … … … … … … … … …

$$\mu_{\underset{\sim}{R}}{}^{(1)}(x_6) = \bigvee_y \mu_{\underset{\sim}{R}}(x_6, y) = MAX\{0.9, \quad 0, \;\; 0.3, \quad 0.7\} = 0.9.$$

One may similarly calculate the second projection. The results are given in Figure 11.2. We see that this relation $\underset{\sim}{R}$ is normal.

***Example 2*:** We consider the case of a relation $x\underset{\sim}{R}y$ where $x \in R^+ \; and \; y \in R^+$ with

$$\mu_{\underset{\sim}{R}}(x, y) = e^{-k(y-x)^2}, k > 1$$

Figure (11.3) which one may take to be defined by the fuzzy phrase: x and y are very near to one another (for a sufficient value of k).

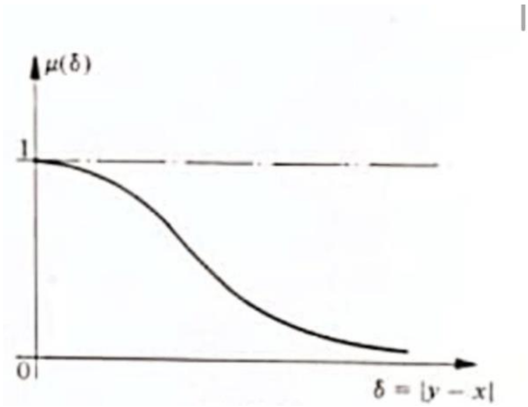


Figure 11.3

In this case we see, for a fixed value x_0,

$$\mu_{\underset{\sim}{R}}{}^{(1)}(x_0) = \bigvee_y \mu_{\underset{\sim}{R}}(x_0, y)$$

$$= \bigvee_y e^{-k(y-x_0)^2}$$

$$= e^{-k(y-x_0)^2} \quad for\ y = x_0$$
$$= 1.$$

One will find the same value for $\mu_{\underset{\sim}{R}}{}^{(1)}(y_0)$ and therefore $h\left(\underset{\sim}{R}\right) = 1$.

Support of a fuzzy relation

One will call the support of $\underset{\sim}{R}$ the ordinary subset of ordered pairs (x, y) for which the membership function is nonzero. Thus

$$S\left(\underset{\sim}{R}\right) = \{(x, y)/\mu_{\underset{\sim}{R}}(x, y) > 0\}.$$

Example 1:

	y_1	y_2	y_3	y_4
x_1	0.1	0	0.2	0
x_2	0.3	0	0	0.9
x_3	0.4	0.7	1	1

Figure 11.4

$$S\left(\underset{\sim}{R}\right) = \{(x_1, y_1), (x_1, y_3), (x_2, y_1), (x_2, y_4)$$
$$(x_3, y_1), (x_3, y_2), (x_3, y_3), (x_3, y_4)\}.$$

Example 2 :

Consider a relation $x\ \underset{\sim}{R}\ y$ where $x \in R^+\ and\ y \in R^+$ with

$$\mu_{\underset{\sim}{R}}(x, y) = e^{-(y-x)^2}, |y - x| \leq 0.46,$$
$$= 0, \qquad |y - x| > 0.46.$$

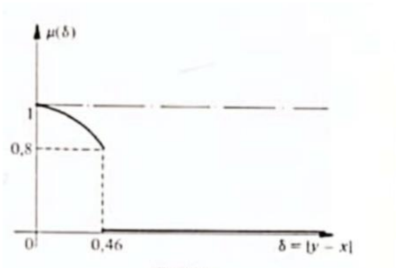


Figure 11.5

One then has

$$S\left(\underset{\sim}{R}\right) = \{\frac{x, y}{|x - y|} < 0.46\}.$$

Envelope of a fuzzy relation:

Let $\underset{\sim}{R}$ and $\underset{\sim}{\varrho}$ be two fuzzy relations such that

$$\forall\ (x, y) \in E_1 \times E_2: \mu_{\underset{\sim}{R}}(x, y) < \mu_{\underset{\sim}{\varrho}}(x, y)$$

one that says that $\underset{\sim}{\varrho}$ is an envelope of $\underset{\sim}{R}$ or that $\underset{\sim}{R}$ is the enclosure of $\underset{\sim}{\varrho}$.

We note

$$\underset{\sim}{R} \subset \underset{\sim}{\varrho}$$

If $\underset{\sim}{\varrho}$ is an envelope of $\underset{\sim}{R}$.

Example 1:

Figure 11.6. One may verify that $\underset{\sim}{\varrho}$ is an envelope of $\underset{\sim}{R}$.

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,3	0,4	0,2	0
x_2	0,5	0	1	0,9
x_3	0,4	0	0,1	0,8

(1)

$\underset{\sim}{\varrho}$	y_1	y_2	y_3	y_4
x_1	0,4	0,4	0,2	0,1
x_2	0,5	0	1	1
x_3	0,5	0,1	0,2	0,9

(2)

Figure 11.6

Example 2:

Consider the fuzzy relation $x \underset{\sim}{R_1} y$ with $x \in R^+$ *and* $y \in R^+$ such that $y \succ x$, that is, “y is much larger than x” expressed by

$$\mu_{\underset{\sim}{R_1}}(x, y) = 0\ , y - x < 0$$

$$= 1 - e^{-k_1(y-x)^2} \ \ {,y-x \geq 0}$$

Let now $k_2 > k_1$; then

$$\mu_{\underset{\sim}{R_2}}(x, y) = 0\ , y - x < 0$$

$$= 1 - e^{-k_1(y-x)^2}\ , y - x \geq 0.$$

is an envelope.

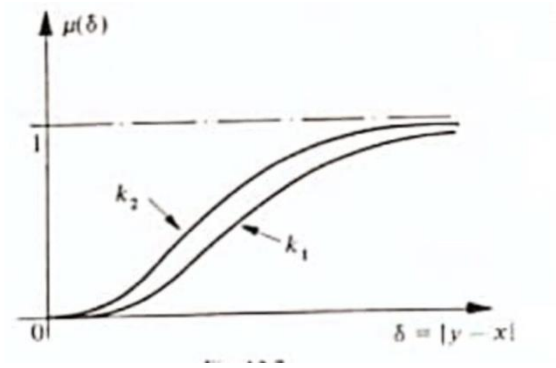


Figure 11.7

Union of two relations:

The Union of two relations $\underset{\sim}{R}$ and $\underset{\sim}{\varrho}$, denoted by $\underset{\sim}{R} \cup \underset{\sim}{\varrho}$ or $\underset{\sim}{R} + \underset{\sim}{\varrho}$, is a relation such that

$$\mu_{\underset{\sim}{R} \cup \underset{\sim}{\varrho}}(x,y) = \mu_{\underset{\sim}{R}}(x,y) \vee \mu_{\underset{\sim}{\varrho}}(x,y)$$

$$= MAX \left\{\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{\varrho}}(x,y)\right\}.$$

If $\underset{\sim}{R_1}, \underset{\sim}{R_2}, \dots \dots \dots \dots \dots, \underset{\sim}{R_n}$ are relations,

$$\mu_{\underset{\sim}{R_1} \cup \underset{\sim}{R_2} \cup \dots\dots\dots\dots\dots \cup \underset{\sim}{R_n}}(x,y) = \bigvee_{\underset{\sim}{R_i}} \mu_{\underset{\sim}{R_i}}(x,y).$$

We note the result,

$$\underset{\sim}{R} = \bigcup_i \underset{\sim}{R_i} \ or \ \sum_i \underset{\sim}{R_i} \ .$$

Example 1:

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,3	0,2	1	0
x_2	0,8	1	0	0,2
x_3	0,5	0	0,4	0

(1)

$\underset{\sim}{Q}$	y_1	y_2	y_3	y_4
x_1	0,3	0	0,7	0
x_2	0,1	0,8	1	1
x_3	0,6	0,9	0,3	0,2

(2)

$\underset{\sim}{R} \cup \underset{\sim}{Q}$	y_1	y_2	y_3	y_4
x_1	0,3	0,2	1	0
x_2	0,8	1	1	1
x_3	0,6	0,9	0,4	0,2

(3)

Figure 11.8

Example 2:

In Figure 11.9(a) we have expressed a fuzzy relation $x \underset{\sim}{R_1} y,\ x \in R^+ \ and\ y \in R^+$, such that "x and y are very near." In Figure 11.9(b) one sees a relation $x \underset{\sim}{R_2} y,\ x \in R^+ \ and\ y \in R^+$, such that "x and y are very different".

The relation $x\ \underset{\sim}{R_3}\, y,\ x \in R^+ \ and\ y \in R^+$, such that "x and y are very near or/and very different" is defined by the curve $\mu_3(x, y)$ such that

$$\begin{aligned} \mu_{\underset{\sim}{R_3}}(x,y) &= 0 \qquad\qquad , |y-x| < 0 \\ &= \mu_{\underset{\sim}{R_1}}(x,y), 0 \le |y-x| \le \propto \\ &= \mu_{\underset{\sim}{R_2}}(x,y), \propto \le |y-x|, \end{aligned}$$

with

$$|y-x| = \propto$$

such that

$$\mu_{\underset{\sim}{R_1}}(x,y) = \mu_{\underset{\sim}{R_2}}(x,y)$$

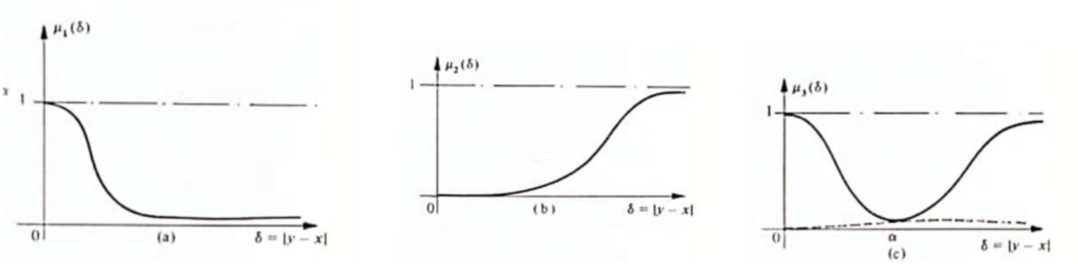


Figure 11.9

In a logic constructed on the theory of ordinary sets, a proposition like "x and y are very near or/and very different must be reduced to "x and y are very near or very different". But with respect to the theory of fuzzy subsets, the first proposition is coherent; it expresses that the "and" case is conceivable with a very weak weight, corresponding to the case where x and y are neither very near nor very different.

This example illustrates well the propositional flexibility that one finds in the present theory.

Intersection of two relations:

The intersection of two relations $\underset{\sim}{R}$ and $\underset{\sim}{\varrho}$, denoted by $\underset{\sim}{R} \cap \underset{\sim}{\varrho}$, is define by the expression

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{\varrho}}(x, y) = \mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{\underset{\sim}{\varrho}}(x, y)$$

$$= MIN \left\{\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{\varrho}}(x, y)\right\}.$$

If $\underset{\sim}{R_1}, \underset{\sim}{R_2}, \dots \dots \dots \dots \dots , \underset{\sim}{R_n}$ are relations,

$$\mu_{\underset{\sim}{R_1} \cap \underset{\sim}{R_2} \cap \dots\dots\dots\dots\dots, \cap \underset{\sim}{R_n}}(x, y) = \bigwedge_{\underset{\sim}{R_i}} \mu_{\underset{\sim}{R_i}}(x, y).$$

We note the result,

$$\underset{\sim}{R} = \bigcap_i \underset{\sim}{R_i}.$$

Example 1: Consider

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,3	0,2	1	0
x_2	0,8	1	0	0,2
x_3	0,5	0	0,4	0

(1)

$\underset{\sim}{\varrho}$	y_1	y_2	y_3	y_4
x_1	0,3	0	0,7	0
x_2	0,1	0,8	1	1
x_3	0,6	0,9	0,3	0,2

(2)

$\underset{\sim}{R} \cap \underset{\sim}{\varrho}$ is,

$\underset{\sim}{R} \cap \underset{\sim}{\varrho}$	y_1	y_2	y_3	y_4
x_1	0,3	0	0,7	0
x_2	0,1	0,8	0	0,2
x_3	0,5	0	0,3	0

Figure 11.10

Example 2:

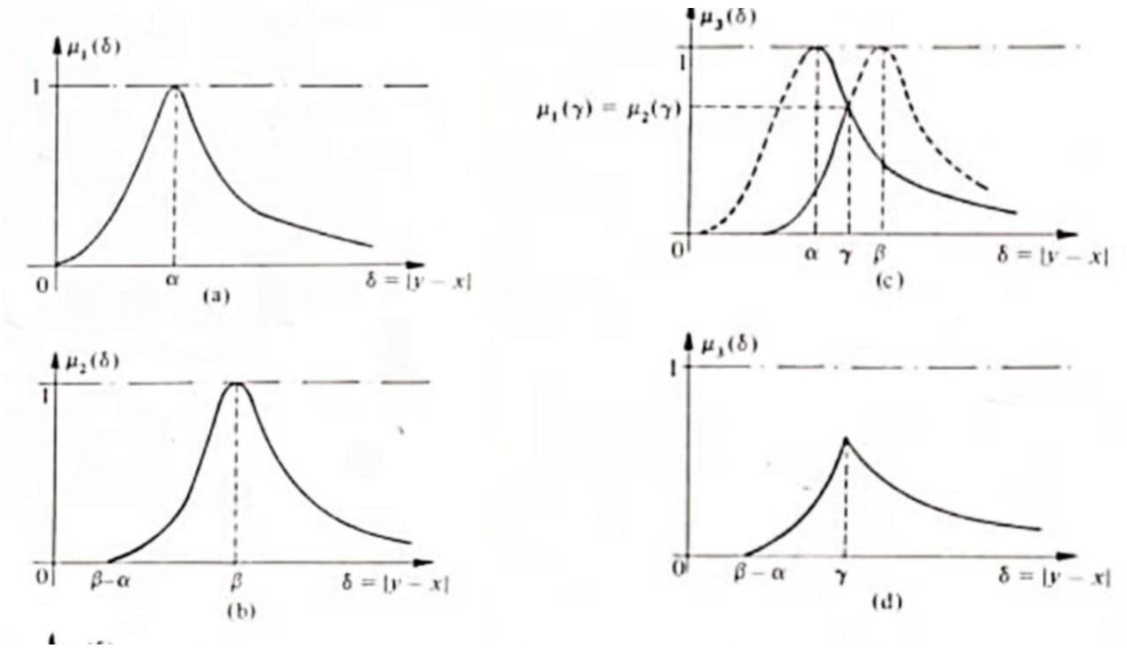


Figure 11.11

In Figure 11.11(a) is expressed the fuzzy relation $x\,\underset{\sim}{R_1}\,y,\ x \in R^+\ and\ y \in R^+$, such that "$|x - y|$ is very near α." In Figure 11.11(b) one sees a relation $x\,\underset{\sim}{R_2}y,\ x \in R^+\ and\ y \in R^+$, such that "$|x - y|$ is very near β" (with $\beta > \alpha$).

Figure 12.11(c) shows how to obtain

$$\underset{\sim}{R_3} = \underset{\sim}{R_1} \cap \underset{\sim}{R_2}$$

One has

$$\begin{aligned} \mu_{\underset{\sim}{R_3}}(x, y) &= 0 \qquad\qquad ,|y - x| < \beta - \alpha \\ &= \mu_{\underset{\sim}{R_1}}(x, y), \qquad \beta - \alpha \le |y - x| \le \gamma \\ &= \mu_{\underset{\sim}{R_2}}(x, y), \qquad \gamma \le |y - x|, \end{aligned}$$

where γ is the value of $|y - x|$ such that $\mu_{\underset{\sim}{R_1}}(x, y) = \mu_{\underset{\sim}{R_2}}(x, y)$. The result appears in Figure 11.11 (d).

Algebraic product of two relations:

One defines the algebraic product of two relations $\underset{\sim}{R}$ and $\underset{\sim}{\varrho}$, denoted by $\underset{\sim}{R} \cdot \underset{\sim}{\varrho}$, is define by the expression

$$\mu_{\underset{\sim}{R}\,\cdot\,\underset{\sim}{\varrho}}(x,y) = \mu_{\underset{\sim}{R}}(x,y)\ \cdot\ \mu_{\underset{\sim}{\varrho}}(x,y).$$

In the right-hand side of this expression, the '$\cdot$' indicates a numerical product (ordinary multiplication).

Example 1:

Consider

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,3	0,2	1	0
x_2	0,8	1	0	0,2
x_3	0,5	0	0,4	0

(1)

$\underset{\sim}{Q}$	y_1	y_2	y_3	y_4
x_1	0,3	0	0,7	0
x_2	0,1	0,8	1	1
x_3	0,6	0,9	0,3	0,2

(2)

Then $\underset{\sim}{R}\ \cdot\ \underset{\sim}{\varrho}$ is

$\underset{\sim}{R}\cdot\underset{\sim}{Q}$	y_1	y_2	y_3	y_4
x_1	0,09	0	0,7	0
x_2	0,08	0,8	0	0,2
x_3	0,3	0	0,12	0

Figure 11.12

Example 2:

Taking again the example considered in Figures 11.11(a) and 11.11(b), let

$$\underset{\sim}{R_3} = \underset{\sim}{R_1} \cdot\ \underset{\sim}{R_2}$$

One has

$$\mu_{\underset{\sim}{R_3}}(x, y) = 0 \qquad\qquad , |y - x| < \beta - \alpha$$

$$= \mu_{\underset{\sim}{R_1}}(x, y) \cdot \mu_{\underset{\sim}{R_2}}(x, y), \qquad \beta - \alpha \leq |y - x|$$

See Figures 11.13 (a)-(c).

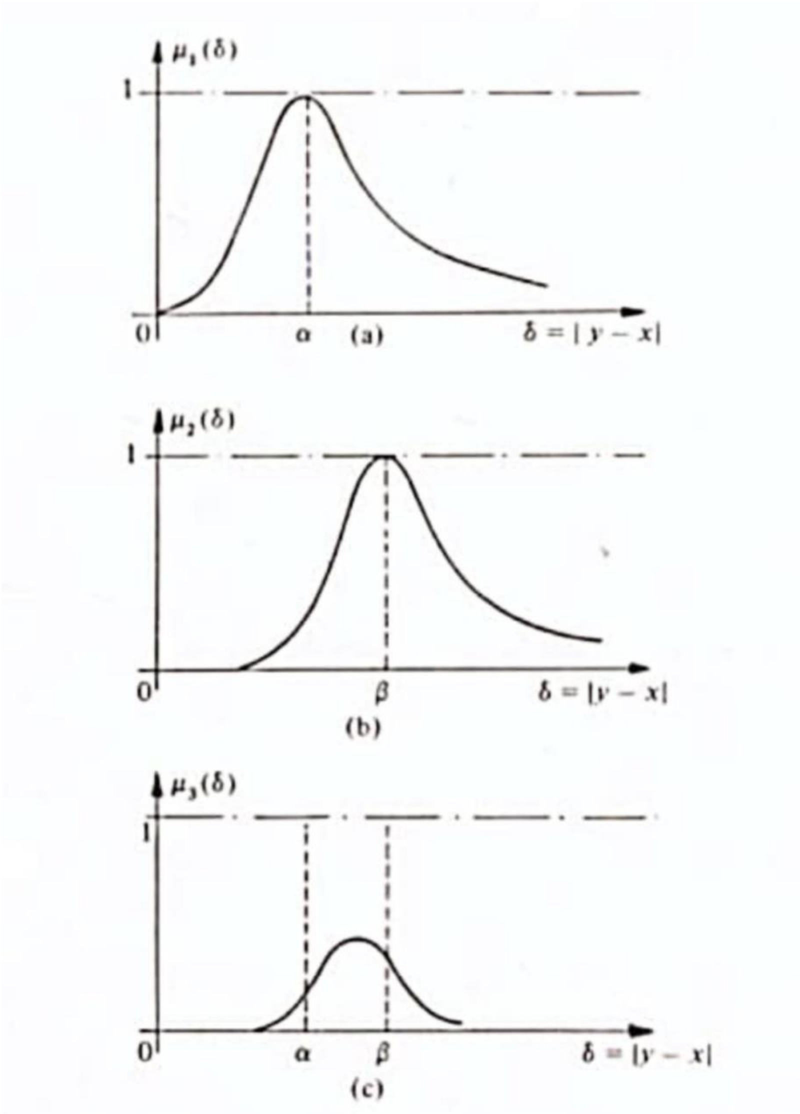


Figure 11.13

Distributivity

We note the properties of distributivity:

$$\underset{\sim}{R} \cap \left(\underset{\sim}{\varrho} \cup \underset{\sim}{\text{б}} \right) = \left(\underset{\sim}{R} \cap \underset{\sim}{\varrho} \right) \cup \left(\underset{\sim}{R} \cap \underset{\sim}{\text{б}} \right),$$

$$\underset{\sim}{R} \cup \left(\underset{\sim}{\varrho} \cap \underset{\sim}{б} \right) = \left(\underset{\sim}{R} \cup \underset{\sim}{\varrho} \right) \cap \left(\underset{\sim}{R} \cup \underset{\sim}{б} \right)$$

$$\underset{\sim}{R} \cdot \left(\underset{\sim}{\varrho} \cup \underset{\sim}{б} \right) = \left(\underset{\sim}{R} \cdot \underset{\sim}{\varrho} \right) \cup \left(\underset{\sim}{R} \cdot \underset{\sim}{б} \right),$$

$$\underset{\sim}{R} \cdot \left(\underset{\sim}{\varrho} \cap \underset{\sim}{б} \right) = \left(\underset{\sim}{R} \cdot \underset{\sim}{\varrho} \right) \cap \left(\underset{\sim}{R} \cdot \underset{\sim}{б} \right),$$

Algebraic sum of two relation

One defines the algebraic sum of two $\underset{\sim}{R}$ and $\underset{\sim}{\varrho}$, denoted $\underset{\sim}{R} \tilde{+} \underset{\sim}{\varrho}$, by the expression

$$\mu_{\underset{\sim}{R} \tilde{+} \underset{\sim}{\varrho}}(x,y) = \mu_{\underset{\sim}{R}}(x,y) + \mu_{\underset{\sim}{\varrho}}(x,y) - \mu_{\underset{\sim}{R}}(x,y) \cdot \mu_{\underset{\sim}{\varrho}}(x,y)$$

The $\cdot$ indicates ordinary multiplication and the sign +, ordinary addition.

Example:

Consider

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,3	0,2	1	0
x_2	0,8	1	0	0,2
x_3	0,5	0	0,4	0

(1)

$\underset{\sim}{\varrho}$	y_1	y_2	y_3	y_4
x_1	0,3	0	0,7	0
x_2	0,1	0,8	1	1
x_3	0,6	0,9	0,3	0,2

(2)

Then $\underset{\sim}{R} \tilde{+} \underset{\sim}{\varrho}$ is

$\underset{\sim}{R} \hat{+} \underset{\sim}{Q}$	y_1	y_2	y_3	y_4
x_1	0,51	0,20	1	0
x_2	0,82	1	1	1
x_3	0,80	0,90	0,58	0,20

Figure 11.14

We note two properties of distributivity:

$$\underset{\sim}{R} \,\tilde{+}\, \left(\underset{\sim}{\varrho} \cup \underset{\sim}{\text{б}}\right) = \left(\underset{\sim}{R} \,\tilde{+}\, \underset{\sim}{\varrho}\right) \cup \left(\underset{\sim}{R} \,\tilde{+}\, \underset{\sim}{\text{б}}\right),$$

$$\underset{\sim}{R} \,\tilde{+}\, \left(\underset{\sim}{\varrho} \cap \underset{\sim}{\text{б}}\right) = \left(\underset{\sim}{R} \,\tilde{+}\, \underset{\sim}{\varrho}\right) \cap \left(\underset{\sim}{R} \,\tilde{+}\, \underset{\sim}{\text{б}}\right).$$

Complement of a relation:

The complement of $\underset{\sim}{R}$, denoted $\underset{\sim}{\bar{R}}$, is the relation such that

$$\forall\ (x, y) \in E_1 \times E_2 \colon \mu_{\underset{\sim}{\bar{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y).$$

Example 1:

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,3	0,4	0,2	0
x_2	0,5	0	1	0,9
x_3	0,4	0	0,1	0,8

(1)

$\underset{\sim}{\bar{R}}$	y_1	y_2	y_3	y_4
x_1	0,7	0,6	0,8	1
x_2	0,5	1	0	0,1
x_3	0,6	1	0,9	0,2

(2)

Figure 11.15

Example 2 :

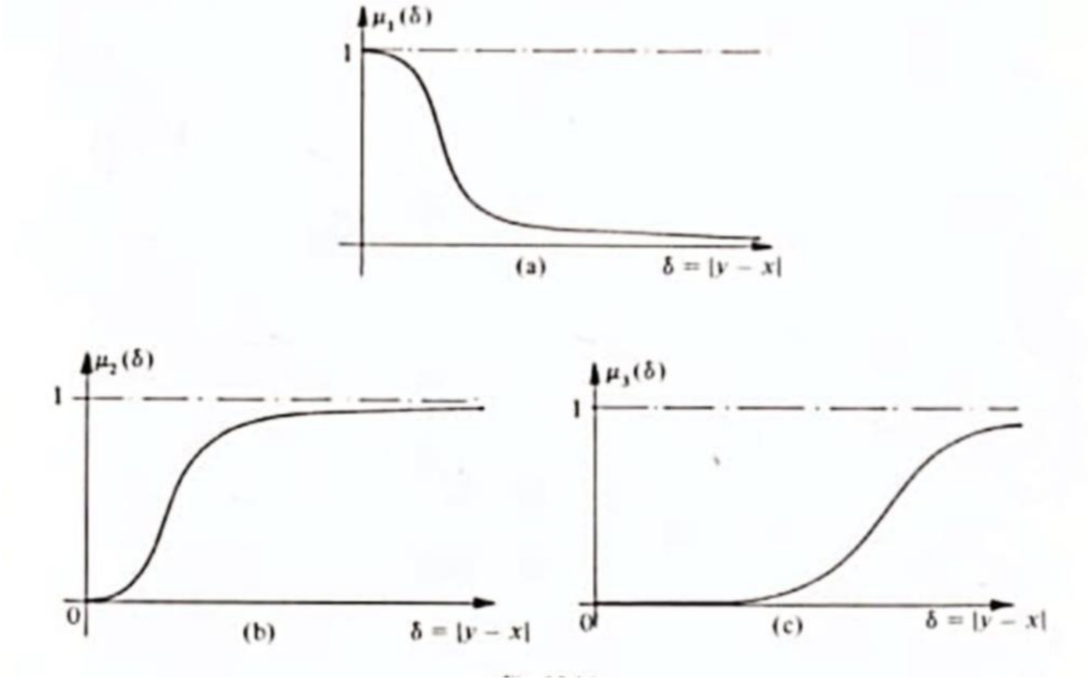


Figure 11.16

In Figure 11.16 (a) is represented the membership function $\mu_{\underset{\sim}{R_1}}(x, y)$, corresponding to the relation $x\underset{\sim}{R_1}y$ signifying "x and y are very near to one another", R^+ and $y \in R^+$.

Figure 11.16 (b) then represents the membership function

$$\mu_{\underset{\sim}{R_2}}(x, y) = 1 - \mu_{\underset{\sim}{R_1}}(x, y),$$

which may be associated with the relation "x and y are not very near".

Figure 11.16 (c) may be taken to represent a membership function $\mu_{\underset{\sim}{R_3}}(x, y)$ relative to the relation "x and y are very difficult from one another".

We note that the two propositions "x and y are not very near" and "x and y are very different" are not generally identical, unless one chooses membership functions that represent both propositions rather poorly.

Disjunctive sum of two relations:

The disjunctive sum, denoted $\underset{\sim}{R} \oplus \underset{\sim}{\varrho}$, is defined by the expression

$$\underset{\sim}{R} \oplus \underset{\sim}{\varrho} = \left(\underset{\sim}{R} \cap \underset{\sim}{\varrho}\right) \cup \left(\underset{\sim}{R} \cap \underset{\sim}{б}\right).$$

Example 1:

$\underset{\sim}{\mathcal{R}}$	y_1	y_2	y_3
x_1	1	0,3	0
x_2	0,1	1	0,8

$\underset{\sim}{\mathcal{Q}}$	y_1	y_2	y_3
x_1	0,2	0,6	1
x_2	0,8	0	0,2

$\bar{\underset{\sim}{\mathcal{R}}}$	y_1	y_2	y_3
x_1	0	0,7	1
x_2	0,9	0	0,2

$\bar{\underset{\sim}{\mathcal{Q}}}$	y_1	y_2	y_3
x_1	0,8	0,4	0
x_2	0,2	1	0,8

$\underset{\sim}{\mathcal{R}} \cap \bar{\underset{\sim}{\mathcal{Q}}}$	y_1	y_2	y_3
x_1	0,8	0,3	0
x_2	0,1	1	0,8

$\bar{\underset{\sim}{\mathcal{R}}} \cap \underset{\sim}{\mathcal{Q}}$	y_1	y_2	y_3
x_1	0	0,6	1
x_2	0,8	0	0,2

$\underset{\sim}{\mathcal{R}} \oplus \underset{\sim}{\mathcal{Q}}$	y_1	y_2	y_3
x_1	0,8	0,6	1
x_2	0,8	1	0,8

Figure 11.17

Example 2:

Consider again the example given in Figures 11.11 (a) and 11.11 (b): Let $\underset{\sim}{R}$ be the relation induced by the membership function in Figure 11.11(a) and $\underset{\sim}{\varrho}$ that pertaining to Figure 11.11(b). By following Figures 11.18 (a) – (i), the reader may see how to obtain the membership function relative to the relation $\underset{\sim}{R} \oplus \underset{\sim}{\varrho}$.

Compare Figures 11.11 (d) and 11.18(i); as may be seen, the disjunctive or Figure 11.18(i) gives a considerably different result than and also rather different than that of or/and (Figure 11.18(j)).

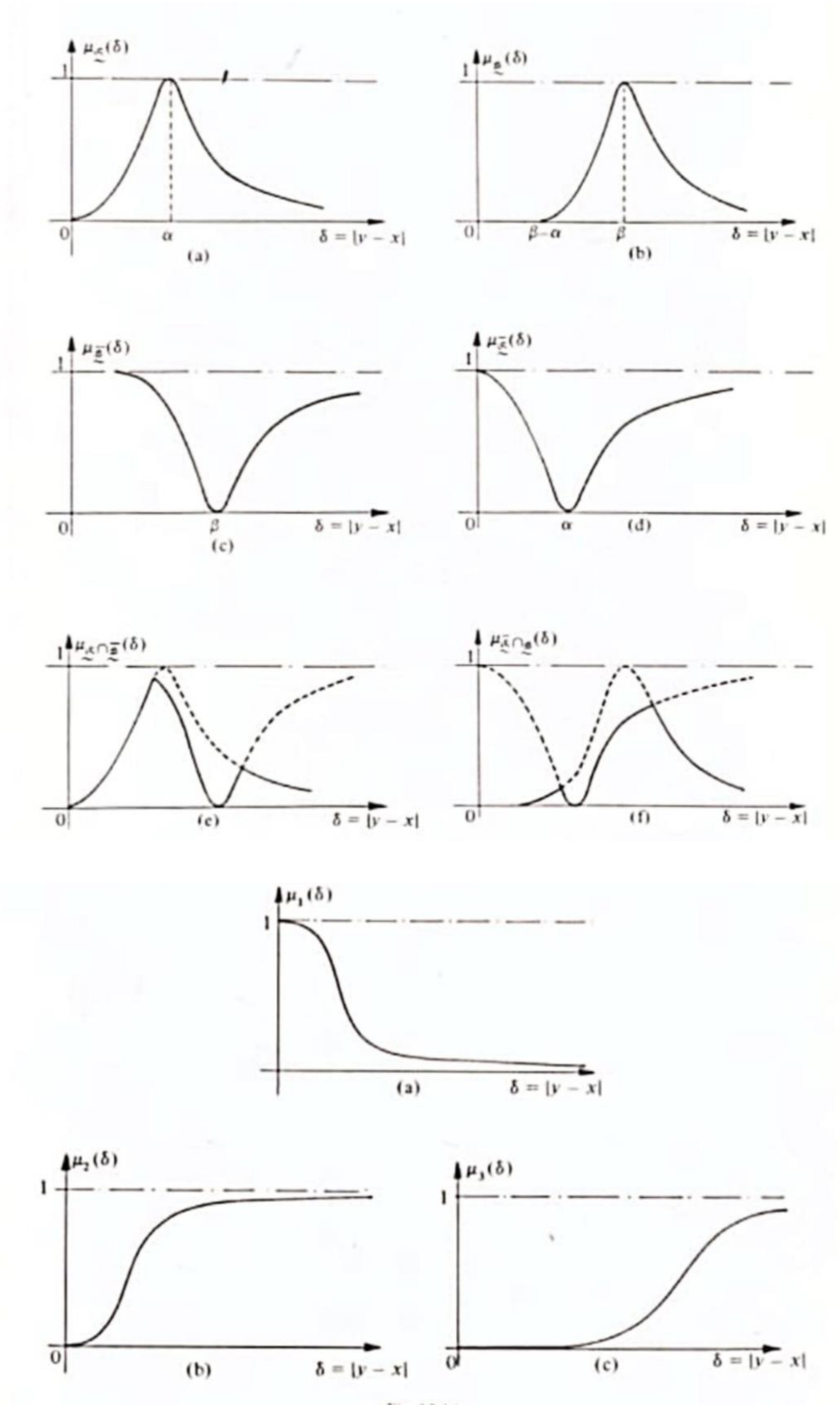


Figure 11.18

One likewise defines the operation of complementation:

$$\underset{\sim}{R}\ \overline{\oplus}\underset{\sim}{\varrho} = \overline{\underset{\sim}{R}\ \oplus\underset{\sim}{\varrho}}$$

$$= \left(\underset{\sim}{R} \cap \underset{\sim}{\bar{Q}}\right) \cup \left(\underset{\sim}{\bar{R}} \cap \underset{\sim}{Q}\right).$$

$\underset{\sim}{R} \,\bar{\oplus}\, \underset{\approx}{Q}$	y_1	y_2	y_3
x_1	0,2	0,4	0
x_2	0,2	0	0,2

Figure 11.19

We reconsider the preceding examples in Figures 11.19 and 11.20. Figure 11.20 has been obtained with reference to Figure 11.18 (i).

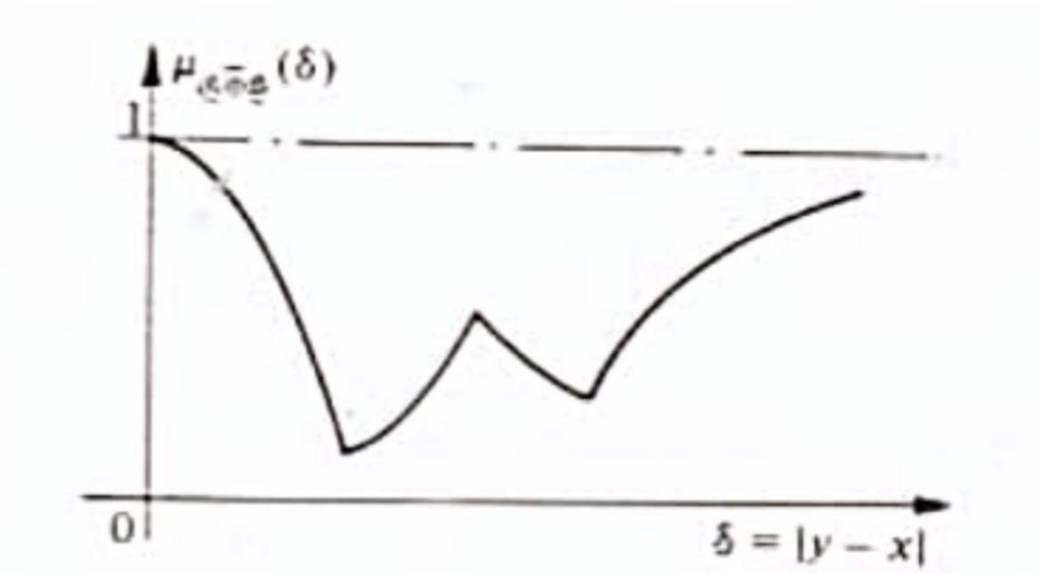


Figure 11.20

Ordinary relation closest to a fuzzy relation:

Let $\underset{\sim}{R}$ be a fuzzy relation; an ordinary relation closest to $\underset{\sim}{R}$ will be given by

$$\mu_{\underset{\approx}{R}}(x,y) = 0 \; if \; \mu_{\underset{\sim}{R}}(x,y) < 0.5$$

$$= 1 \; if \; \mu_{\underset{\sim}{R}}(x, y) > 0.5$$

$$= 0 \; or \; 1 \; if \; \mu_{\underset{\sim}{R}}(x, y) = 0.5$$

Example :

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4	y_5	y_6
x_1	0,7	0,3	0,2	1	0	0,8
x_2	0,5	0,4	0	0,6	0,9	0,1
x_3	0,6	1	0,8	0	0	0,7

Figure 11.21

$\underset{\approx}{R}$	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1	0	0	1	0	1
x_2	0	0	0	1	1	0
x_3	1	1	1	0	0	1

Figure 11.22

12. COMPOSITION OF TWO FUZZY RELATIONS

We mention now that sometimes we will use the notion

$$\underset{\sim}{R} \subset X \times Y$$

signifying

$$\underset{\sim}{G} \subset X \times Y$$

where $\underset{\sim}{R}$ is the fuzzy relation corresponding to the fuzzy graph $\underset{\sim}{G}$.

Max-Min composition:

Let $\underset{\sim}{R_1} \subset X \times Y$ and $\underset{\sim}{R_2} \subset X \times Y$. We define the min-max composition of $\underset{\sim}{R_1} \; and \; \underset{\sim}{R_2}$, denoted by $\underset{\sim}{R_1} \circ \underset{\sim}{R_2}$, by the expression

$$\mu_{\underset{\sim}{R_1} \circ \underset{\sim}{R_2}}(x, z) = \bigvee_{y} [\, \mu_{\underset{\sim}{R_1}}(x, y) \wedge \mu_{\underset{\sim}{R_2}}(x, y)]$$

$$= \underset{y}{MAX}[\, MIN(\, \mu_{\underset{\sim}{R_1}}(x, y), \mu_{\underset{\sim}{R_2}}(x, y))],$$

where $x \in X, y \in Y$, and z$\in Z$.

Example 1:

Consider two fuzzy relations $x\underset{\sim}{R_1}y$ and $y\underset{\sim}{R_2}z$ $, where\ x, y\ and\ z \in R^+$. We suppose

$$\mu_{\underset{\sim}{R_1}}(x,y) = e^{-k(x-y)^2}, k \geq 1.$$

$$\mu_{\underset{\sim}{R_2}}(y,z) = e^{-k(y-z)^2}, k \geq 1.$$

We now propose to determine $\mu_{\underset{\sim}{R_1} \circ \underset{\sim}{R_2}}(x,z)$.

Consider two values x=a and z=b of the variables x and z. Here the membership functions are continuous on the interval $[0,\infty)$; we may write

$$\mu_{\underset{\sim}{R_1} \circ \underset{\sim}{R_2}}(a,b) = \bigvee_y [\,\mu_{\underset{\sim}{R_1}}(a,y) \wedge \mu_{\underset{\sim}{R_2}}(y,b)]$$

$$= \bigvee_y [\,e^{-k(a-y)^2} \wedge e^{-k(y-b)^2}]\,.$$

The composition of $\underset{\sim}{R_1}$ and $\underset{\sim}{R_2}$ with respect to the max-min operation is represented in Figure 12.1. Thus,

$$\mu_{\underset{\sim}{R_1} \circ \underset{\sim}{R_2}}(a,b) = e^{-k\left(a-\frac{a+b}{2}\right)^2}$$

$$= e^{-k\left(\frac{a-b}{2}\right)^2}.$$

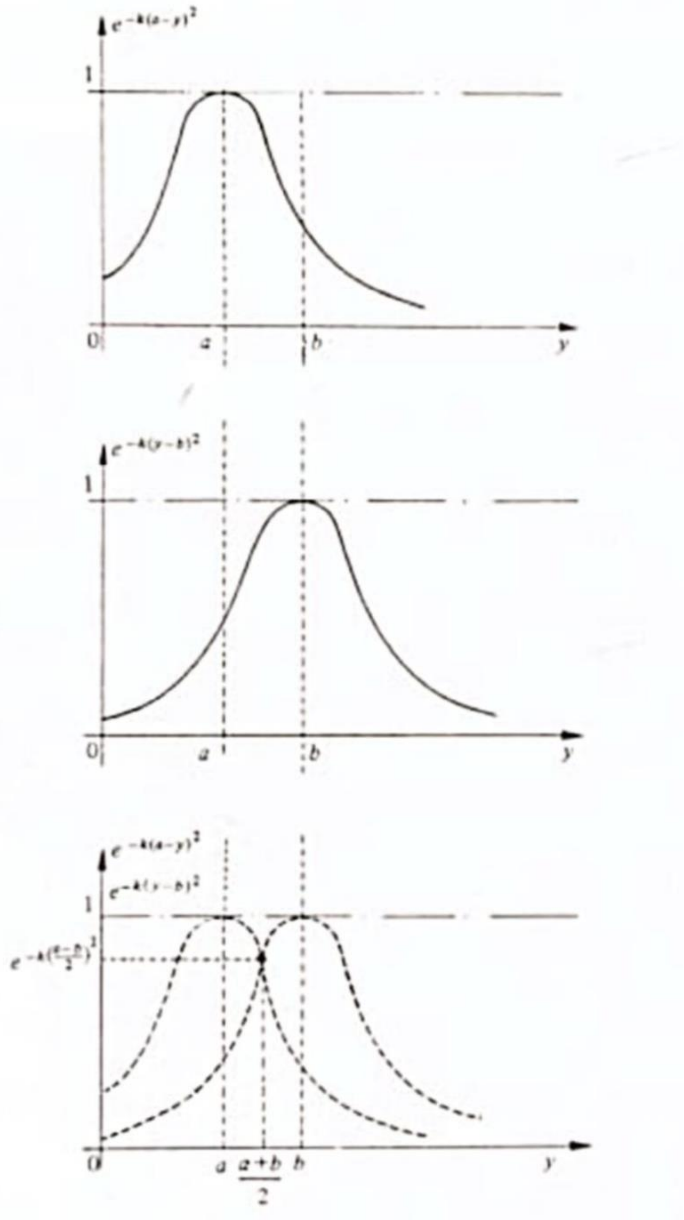


Figure 12.1

and, for all values of x and z,

$$\mu_{\underset{\sim}{R_1} \circ \underset{\sim}{R_2}}(x,z) = e^{-k\left(\frac{x-z}{2}\right)^2}.$$

For simplicity, we have here considered two identical functions $\mu_{\underset{\sim}{R_1}}(x,y)$ and $\mu_{\underset{\sim}{R_1}}(y,z)$, but the reasoning remains the same for two distinct functions.

Example 2: (Figure 12.2)

$\underset{\sim}{R}_1$	y_1	y_2	y_3	y_4	y_5
x_1	0,1	0,2	0	1	0,7
x_2	0,3	0,5	0	0,2	1
x_3	0,8	0	1	0,4	0,3

(a)

$\underset{\sim}{R}_2$	z_1	z_2	z_3	z_4
y_1	0,9	0	0,3	0,4
y_2	0,2	1	0,8	0
y_3	0,8	0	0,7	1
y_4	0,4	0,2	0,3	0
y_5	0	1	0	0,8

(b)

$\underset{\sim}{R}_2 \circ \underset{\sim}{R}_1$	z_1	z_2	z_3	z_4
x_1	0,4	0,7	0,3	0,7
x_2	0,3	1	0,5	0,8
x_3	0,8	0,3	0,7	1

(c)

Figure 12.2

Let $(x, z) = (x_1, z_1)$

$$MIN\left(\mu_{\underset{\sim}{R_1}}(x_1, y_1), \mu_{\underset{\sim}{R_2}}(y_1, z_1)\right) = MIN(0.1, 0.9)$$

$$= 0.1.$$

$$MIN\left(\mu_{\underset{\sim}{R_1}}(x_1, y_2), \mu_{\underset{\sim}{R_2}}(y_2, z_1)\right) = MIN(0.2, 0.2)$$

$$= 0.2.$$

$$MIN\left(\mu_{\underset{\sim}{R_1}}(x_1, y_3), \mu_{\underset{\sim}{R_2}}(y_3, z_1)\right) = MIN(0, 0.8) = 0$$

$$MIN\left(\mu_{\underset{\sim}{R_1}}(x_1, y_4), \mu_{\underset{\sim}{R_2}}(y_4, z_1)\right) = MIN(1, 0.4) = 0.4$$

$$MIN\left(\mu_{\underset{\sim}{R_1}}(x_1, y_5), \mu_{\underset{\sim}{R_2}}(y_5, z_1)\right) = MIN(0.7, 0) = 0$$

$$\underset{y_i}{MAX}[\,MIN(\,\mu_{\underset{\sim}{R_1}}(x_i, y_i), \mu_{\underset{\sim}{R_2}}(y_i, z_i))]$$

$$= MAX\{0.1, 0.2, 0, 0.4, 0\} = 0.4.$$

Now let $(x, z) = (x_1, z_2)$

$$MIN\left(\mu_{\underset{\sim}{R_1}}(x_1, y_1), \mu_{\underset{\sim}{R_2}}(y_1, z_2)\right) = MIN(0.1, 0) = 0,$$

$$MIN\left(\mu_{\underset{\sim}{R_1}}(x_1, y_2), \mu_{\underset{\sim}{R_2}}(y_2, z_2)\right) = MIN(0.2, 1) = 0.2, \dots \dots ..$$

and so on. The results are given in Figure 12.2.

Example 3:

In Figure 12.3 an example of the composition of three relations is presented.

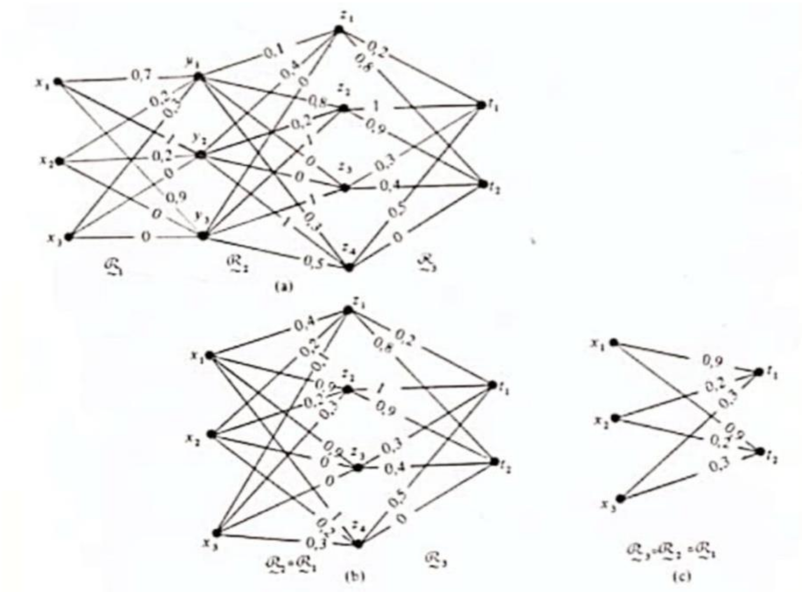


Figure 12.3

The max-min composition operation is associative, that is,

$$(\underset{\sim}{R_3} \circ \underset{\sim}{R_2})\circ \underset{\sim}{R_1} = \underset{\sim}{R_3} \circ (\underset{\sim}{R_2}\circ \underset{\sim}{R_1}).$$

On the other hand, if $\underset{\sim}{R}$ is a relation defined on $E \times E$, then $\underset{\sim}{R} \subset E \times E$; one may write

$$\underset{\sim}{R}\circ\underset{\sim}{R} = \underset{\sim}{R}^2.$$

and from this

$$\underset{\sim}{R}\circ\underset{\sim}{R}^2 = \underset{\sim}{R}^2 \circ \underset{\sim}{R} = \underset{\sim}{R}^3;$$

and more generally

$$\underbrace{\underset{\sim}{R}\circ\underset{\sim}{R}\circ\underset{\sim}{R} \ldots\ldots\ldots\ldots\circ\underset{\sim}{R}}_{\text{k times}} = \underset{\sim}{R}^k.$$

Max-star composition:

If we replace the operation $\wedge$ in

$$\mu_{\underset{\sim}{R_1}\circ\underset{\sim}{R_2}}(x,z) = \bigvee_y [\mu_{\underset{\sim}{R_1}}(x,y)\wedge\mu_{\underset{\sim}{R_2}}(y,z)]$$

arbitrarily with another, under the restriction that one uses an operation, like $\wedge$, that is associative and monotone nondecreasing in each argument. One may then write

$$\mu_{\underset{\sim}{R_1}\circ\underset{\sim}{R_2}}(x,z) = \bigvee_y [\mu_{\underset{\sim}{R_1}}(x,y)\circ\mu_{\underset{\sim}{R_2}}(y,z)].$$

Max-product composition:

Among the max-star compositions that may be imagined, the max-product composition deserves our particular attention. In this case, the operation $\circ$ will be the product designated by $\cdot$, the formula then becomes

$$\mu_{\underset{\sim}{R_1}\cdot\underset{\sim}{R_2}}(x,z) = \bigvee_y [\mu_{\underset{\sim}{R_1}}(x,y)\cdot\mu_{\underset{\sim}{R_2}}(y,z)].$$

Ordinary subset of level α in fuzzy relation

Let $\alpha \in [0,1]$; we shall call the ordinary subset of level α of a fuzzy relation $\underset{\sim}{R} \subset X \times Y$, the ordinary subset

$$G_\alpha = \{(x,y) |\ \mu_{\underset{\sim}{R}}(x,y) \geq \alpha\}.$$

Example 1: (Figure 12.4)

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4
x_1	0,3	0,8	1	0
x_2	0,5	1	0,3	0,9
x_3	1	0,2	0,6	0,7

Figure 12.4

$$G_{0.8} = \{\, (x_1, y_2),\ (x_1, y_3),\ (x_2, y_2),\ (x_2, y_4),\ (x_3, y_1)\}.$$

Example 2:

Consider the fuzzy relation defined in R^2 by

$$\mu_{\underset{\sim}{R}}(x, y) = 1 - \frac{1}{1 + x^2 + y^2}.$$

The subset of level 0.3 will be defined by

$$1 - \frac{1}{1 + x^2 + y^2} \geq 0.3$$

Thus,

$$x^2 + y^2 \geq \frac{3}{7}.$$

This subset is the exterior and circumference of the circle with radius $r = \sqrt{3/7}$. (See Figure 12.5).

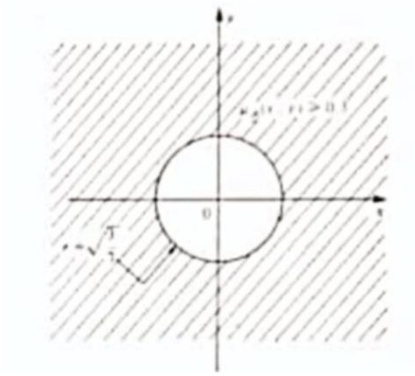


Figure 12.5

Important Property:

One now has the evident property

$$\alpha_1 \geq \alpha_2 \Rightarrow G_{\alpha_1} \subset G_{\alpha_2}.$$

Or, what is the same thing,

$$R_{\alpha_1} \subset R_{\alpha_2}.$$

Decomposition Theorem:

Any fuzzy relation $\underset{\sim}{R}$ may be decomposed in the form

$$\underset{\sim}{R} = \bigvee_{\alpha} \alpha \cdot R_{\alpha}, 0 < \alpha \leq 1$$

where

$$\mu_{\underset{\sim}{R_\alpha}}(x, y) = 1 \; if \;\; \mu_{\underset{\sim}{R}}(x, y) \geq \alpha$$

$$= 0 \; if \; \mu_{\underset{\sim}{R}}(x, y) < \alpha$$

Here $\alpha\ R_\alpha$ indicates that all the elements of the ordinary relation R_α are multiplied by α.

Proof:

The membership function of

$$\underset{\sim}{R} = \bigvee_{\alpha} \alpha \cdot R_{\alpha}, 0 < \alpha \leq 1$$

may be written as $\mu_{\vee_\alpha \alpha \cdot R_\alpha} = \vee_\alpha \alpha\, \mu_{\underset{\sim}{R_\alpha}}(x, y)$

$$= \bigvee_{\alpha \leq \mu_{\underset{\sim}{R}}(x,y)} \alpha$$

$$= \mu_{\underset{\sim}{R}}(x, y).$$

13. FUZZY SUBSET INDUCED BY A MAPPING

Consider a mapping of set E_1 into a set E_2, denote

$$E_1 \underset{\Gamma}{\overset{\rightsquigarrow}{}} E_2$$

where, if $x \in E_1$ and y $\in E_2$,

$$y \in \boldsymbol{\Gamma}\{\boldsymbol{x}\}$$

Let $\mu_{\underset{\sim}{A}}(x)$ be the membership function of a fuzzy subset $\underset{\sim}{A} \subset E_1$; then the mapping Γ induces E_2 a fuzzy subset $\underset{\sim}{B} \subset E_2$ whose membership function is

$$\mu_{\underset{\sim}{B}}(y) = \max_{x \in \Gamma^{-1}(y)} \{\mu_{\underset{\sim}{A}}(x)\}, if \;\; \Gamma^{-1}(y) \neq \emptyset$$

$$= 0 \qquad\qquad , if \;\; \Gamma^{-1}(y) = \emptyset$$

Example 1:

Let $E_1 = \{x_1, x_2, x_3, x_4,\ x_5,\ x_6,\ x_7\}$ and $E_2 = \{y_1, y_2, y_3, y_4\}$.

Consider a mapping such that

$$\Gamma\{x_1\} = \{y_2\},$$
$$\Gamma\{x_2\} = \{y_1, y_4\},$$
$$\Gamma\{x_3\} = \{y_1\},$$
$$\Gamma\{x_4\} = \{y_3\},$$
$$\Gamma\{x_5\} = \{y_1\},$$
$$\Gamma\{x_6\} = \{y_2\},$$
$$\Gamma\{x_7\} = \{y_4\}.$$

Also consider the inverse mapping Γ^{-1}:

$$\Gamma^{-1}(y_1) = \{x_2, x_3,\ x_5\},$$

$$\Gamma^{-1}(y_2) = \{x_1, x_6\},$$

$$\Gamma^{-1}(y_3) = \{x_4\},$$

$$\Gamma^{-1}(y_4) = \{x_2, x_7\}.$$

And finally consider the fuzzy subset $\underset{\sim}{A} \subset E_1$:

$$\underset{\sim}{A} = \{(x_1|0.3), (x_2|0.7), (x_3|1), (x_4|0), (x_5|0.2), (x_6|0.9), (x_7|0.8)\}.$$

One then has

$$\mu_{\underset{\sim}{B}}(y_1) = \max_{\{x_2, x_3,\ x_5\}}(0.7; 1; 0.2) = 1,$$

$$\mu_{\underset{\sim}{B}}(y_2) = \max_{\{x_1, x_6\}}(0.3; 0.9) = 0.9,$$

$$\mu_{\underset{\sim}{B}}(y_3) = \max_{\{x_4\}}(0) = 0,$$

$$\mu_{\underset{\sim}{B}}(y_2) = \max_{\{x_2, x_7\}}(0.7; 0.8) = 0.8.$$

These results have been portrayed in Figure 13.1

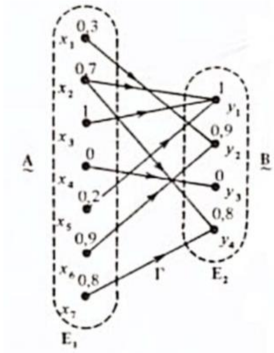


Figure 13.1

It is interesting to compare this notion with the corresponding one for ordinary subsets. Consider Figure 13.2

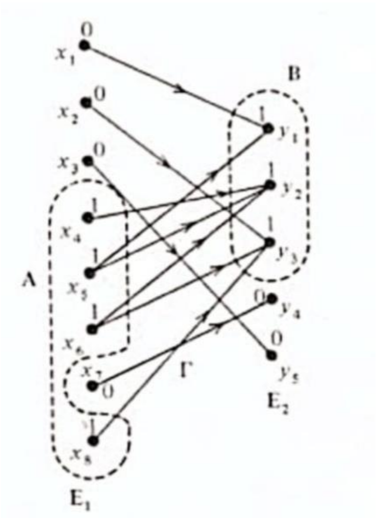


Figure 13.2

Let

$$E_1 = \{x_1, x_2, x_3, x_4,\ x_5,\ x_6,\ \ x_7,\ \ \ x_8\},$$

and

$$E_2 = \{y_1, y_2, y_3, y_4, y_5\}.$$

$$\Gamma\{x_4,\ x_5,\ x_6,\ \ x_8\} = \{y_1, y_2, y_3\}.$$

To the subset $A = \{\ x_4,\ x_5,\ x_6,\ \ x_8\}$, the mapping Γ associates the subset $B = \{y_1, y_2, y_3\}$.

Example 2:

Let $x \in R, y \in R,$ where R is the set of real numbers. We consider the fuzzy subset $\underset{\sim}{A}$ defined by "x near $\frac{(4k+1)\pi}{2}$, $k = \cdots \ldots \ldots .,-2,-1,0,1,2, \ldots \ldots \ldots$". We consider also the function

$$y = f(x) = sinx;$$

then the fuzzy subset $\underset{\sim}{B}$ induced by f(x) will be

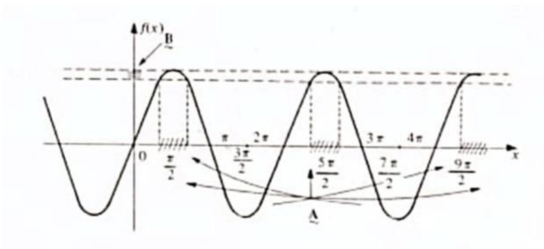


Figure 13.3

14. CONDITIONED FUZZY SUBSETS

A fuzzy subset $\underset{\sim}{B}(x)$ ⊂E_2will be said to be conditioned on E_1 if its membership function depends on x ∈ E_1 as a parameter.

The conditional membership function will then be written

$\mu_{\underset{\sim}{B}}$ (y ‖ x) , where $x \in E_1$ and $y \in E_2$

This function defines a mapping of E_1 into the set of fuzzy subsets defined on E_2. Thus, a fuzzy subset $\underset{\sim}{A} \subset E_1$, will induce a fuzzy subset$\underset{\sim}{B} \subset E_2$, whose membership function will be

$\mu_{\underset{\sim}{B}}$ (y) = $\underset{x\in E_1}{MAX}(MIN\ |\mu_{\underset{\sim}{B}}\,(y||x), \mu_{\underset{\sim}{A}}\,(x))$

We immediately see an example.

Example. Consider a fuzzy relation existing between

$E_1 = \{x_1, x_2, \dots, x_6\}$,

$$E_2 = \{y_1, y_2, y_3\}$$

and defined by

$\underset{\sim}{R}$	y_1	y_2	y_3
x_1	0,3	0,7	0
x_2	0,2	0,5	0
x_3	1	0	0,8
x_4	0	1	0,5
x_5	0,3	1	0,4
x_6	0,8	0	0

This relation $\underset{\sim}{R}$expresses a conditional membership function

$$\mu_{\underset{\sim}{B}}\ (y \,\|\, x)$$

Thus

$$\mu_{\underset{\sim}{B}}\ (y_3 \| \, x_3) = 0.4$$

Suppose that one has a fuzzy subset$\underset{\sim}{A}\ of\ E_1$defined by

$$\underset{\sim}{A}\ = \{(x_1|0.5), (x_2|0.2), (x_3|0.8), (x_4|1), (x_5|0.7)\ , (x_6|\ 0)\},$$

To this fuzzy subset $\underset{\sim}{A} \subset E_1$, corresponds a fuzzy subset in E_2, say $\underset{\sim}{B} \subset E_2$. We carry out the calculations.

First we calculate $\mu_{\underset{\sim}{B}}\,(y_1)$ One has

$$\text{MIN}\,[\mu_{\underset{\sim}{B}}\,(y_1 \parallel x_1)\,, \mu_{\underset{\sim}{A}}\,(x_1\,)]$$

$$= \text{MIN}\,[0.3\,,\,0.5] = 0.3.$$

$$\text{MIN}\,[\mu_{\underset{\sim}{B}}\,(y_1 \parallel x_2)\,, \mu_{\underset{\sim}{A}}\,(x_2\,)]$$

$$= \text{MIN}\,[0.2\,,\,0.2] = 0.2.$$

$$\text{MIN}\,[\mu_{\underset{\sim}{B}}\,(y_1 \parallel x_3)\,, \mu_{\underset{\sim}{A}}\,(x_3\,)]$$

$$= \text{MIN}\,[1\,,\,0.8] = 0.8.$$

$$\text{MIN}\,[\mu_{\underset{\sim}{B}}\,(y_1 \parallel x_4)\,, \mu_{\underset{\sim}{A}}\,(x_4\,)]$$

$$= \text{MIN}\,[0\,,\,1] = 0.$$

$$\text{MIN}\,[\mu_{\underset{\sim}{B}}\,(y_1 \parallel x_5)\,, \mu_{\underset{\sim}{A}}\,(x_5\,)]$$

$$= \text{MIN}\,[0.3\,,\,0.7] = 0.3.$$

$$\text{MIN}\,[\mu_{\underset{\sim}{B}}\,(y_1 \parallel x_6)\,, \mu_{\underset{\sim}{A}}\,(x_6\,)]$$

$$= \text{MIN}\,[0.8\,,\,0] = 0$$

Then

$$\text{MAX MIN}\,[\mu_{\underset{\sim}{B}}\,(y_1 \parallel x_i)\,, \mu_{\underset{\sim}{A}}\,(x_i\,)]$$

$$= \text{MAX}\,[0.3\ \text{o}.2\,,\,0.8\,,\,0\,,\,0.3\,,0]$$

$$= 0.8.$$

One should then do the same for y_2, then y_3. One will finally obtain

$$\mu_{\underset{\sim}{B}}\,(y_1) = 0.8 \quad , \quad \mu_{\underset{\sim}{B}}\,(y_2) = 1\mu_{\underset{\sim}{B}}\,(y_3) = 0.8$$

Thus,

$$\underset{\sim}{B} = \{(y_1|\,0.8)\,, (y_2|1)\,, (y_3|0.8)\,\}$$

Another presentation of this concept. The expression (15.2), as we shall see later, plays for fuzzy subsets the same role as the notion of function for the elements of formal sets. The notion of function for these elements may be expressed by the phrase: "if x = a, then y = b by the function f," which may be written

$$x \quad \underset{f}{\leadsto} \quad y$$

or likewise

$$y = f(x).$$

The notion of conditioned fuzzy subsets plays exactly the same role, but instead of considering elements $x \in E_1$, $y \in E_2$ and the relation f, which is a function, one will make the following definition.

Let $\underset{\sim}{X} \subset E_1$ and $\underset{\sim}{Y} \subset E_2$ consider the fuzzy relation $\underset{\sim}{R}$ existing between E_1 and E_2. One then defines: If $\underset{\sim}{X} = \underset{\sim}{A}$, then $\underset{\sim}{Y} = \underset{\sim}{B}$ through the relation $\underset{\sim}{R}$; this may be written:

$$\underset{\sim}{A}\ ; \quad \underset{\underset{\sim}{R}}{\leadsto} \quad \underset{\sim}{B}\ ;$$

If $\mu_{\underset{\sim}{B}}\,(x, y)$ is the membership function of the fuzzy relation $\underset{\sim}{R}$, $\mu_{\underset{\sim}{A}}$ that of $\underset{\sim}{A}$, and $\mu_{\underset{\sim}{B}}\,(x)$ that of $\underset{\sim}{B}$, one sees that then

$$\mu_{\underset{\sim}{B}}\,(y) = \underset{x \in E_1}{MAX}(MIN\ |\mu_{\underset{\sim}{A}}\,(x), \mu_{\underset{\sim}{A}}\,(x, y))$$

$$= \mathrm{V}\ |\mu_{\underset{\sim}{A}}\,(x) \wedge \mu_{\underset{\sim}{A}}\,(x, y)|$$

This constitutes another presentation of conditioned fuzzy subsets. We consider an example using this presentation.

Example 1.

$$E_1 = \{x_1, x_2, x_3\}$$

$$\underset{\sim}{A} = \{(x_1|0.3), (x_2|0.7)\ , (x_3|1)\}$$

$$E_2 = \{y_1, y_2, y_3\ , y_4\ , y_5\}$$

$\underset{\sim}{R}$	y_1	y_2	y_3	y_4	y_5
x_1	0,8	1	0	0,3	0,7
x_2	0,8	0,3	0,8	0,4	0,7
x_3	0,2	0,3	0	0,2	1

Figure 14.1

We present $\underset{\sim}{A} = \{(x_1|0.3), (x_2|0.7)\ , (x_3|1)\}$ as follows:

$$\underset{\sim}{A} = \begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \multicolumn{1}{|c|}{0,3} & \multicolumn{1}{c|}{0,7} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \; .$$

Figure 14.2

We now carry out the operation min for all the elements of the row with the column y_1 of ; this gives

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline \multicolumn{1}{|c|}{0,3} & \multicolumn{1}{c|}{0,7} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \wedge \begin{array}{c} y_1 \\ \hline \multicolumn{1}{|c|}{0,8} \\ \hline \multicolumn{1}{|c|}{0,8} \\ \hline \multicolumn{1}{|c|}{0,2} \\ \hline \end{array} = \begin{array}{c} y_1 \\ \hline \multicolumn{1}{|c|}{0,3 \wedge 0,8} \\ \hline \multicolumn{1}{|c|}{0,7 \wedge 0,8} \\ \hline \multicolumn{1}{|c|}{1 \wedge 0,2} \\ \hline \end{array} = \begin{array}{c} y_1 \\ \hline \multicolumn{1}{|c|}{0,3} \\ \hline \multicolumn{1}{|c|}{0,7} \\ \hline \multicolumn{1}{|c|}{0,2} \\ \hline \end{array} \; .$$

Figure 14.3

Carrying out the operation max on the elements of the column obtained, we have

$$0,3 \vee 0,7 \vee 0,2 = 0,7.$$

Thus, $\mu_{\underset{\sim}{B}}(y_1) = 0.7$.

Doing the same between the elements of Fig. 14.2 and the other columns of 14.1, we have,

$$\mu_{\underset{\sim}{B}}(y_2) = 0.3 \quad , \mu_{\underset{\sim}{B}}(y_3) = 0.7 \, , \quad \mu_{\underset{\sim}{B}}(y_4) = 0.4, \quad \mu_{\underset{\sim}{B}}(y_5) = 1.$$

And finally

$$\underset{\sim}{B} = \{(y_1|0.7), (y_2|0.3), (y_3|0.7), (y_4|0.4), (y_5|1)\},$$

or what is the same

$$\underset{\sim}{B} = \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \\ \hline \multicolumn{1}{|c|}{0,7} & \multicolumn{1}{c|}{0,3} & \multicolumn{1}{c|}{0,7} & \multicolumn{1}{c|}{0,4} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \; .$$

15. PROPERTIES OF FUZZY BINARY RELATIONS

We shall consider the case where

$$E_1 = E_2 = E \; and \; M = [0,1],$$

and occupy ourselves with some properties of fuzzy binary relations in $E \times E$.

Example 1: Let

$$E = \{A, B, C, D, E\},$$

$$M = \{0, 1\}.$$

$\underset{\sim}{R}$	A	B	C	D	E
A	0,1	0	0	1	0,8
B	0,8	0,3	0	0,7	1
C	0,8	0,3	0,2	0	0,9
D	0,6	0	1	0,5	0
E	0,2	0,5	1	0,6	0,4

Figure 15.1

The table or matrix in Figure 15.1 represents a fuzzy relation in $E \times E$.

Example 2:

Let R be the set of real numbers, and let $x \in R, y \in R$, then

$$|y| \gg |x|$$

is a fuzzy binary relation $\underset{\sim}{R}$ in $R \times R$ provided one is given $\mu_{\underset{\sim}{R}}(x, y)$ defined above, for all (x,y).

Symmetry :

A symmetric fuzzy binary relation is defined by

$$\forall\, (x, y) \in E \times E: \left(\mu_{\underset{\sim}{R}}(x, y) = \mu\right) \Rightarrow \left(\mu_{\underset{\sim}{R}}(y, x) = \mu\right).$$

Example : See Figure 15.2

$\underset{\sim}{R}$	A	B	C	D	E
A	0	0,1	0	0,1	0,9
B	0,1	1	0,2	0,3	0,4
C	0	0,2	0,8	0,8	1
D	0,1	0,3	0,8	0,7	1
E	0,9	0,4	1	1	0

Figure 15.2

Another example : Let R be the set of real numbers, and let $x \in R, y \in R$; then the relation

$$y \text{ is near } x$$

intuitively is a fuzzy symmetric relation in R$\times R$.

Reflexivity : This property is defined by

$$\forall\,(x,x) \in E \times E\colon \mu_{\underset{\sim}{R}}\,(x,x) = 1.$$

Example : See Figure 15.3

$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	1	0	0,2	0,3
B	0	1	0,1	1
C	0,2	0,7	1	0,4
D	0	1	0,4	1

Figure 15.3

Another example : y is near x, in the example given for symmetry is reflexive.

Transitivity :

Let $x, y, z \in E$; then $\forall\,(x,y),(y,z),(x,z) \in E \times E$:

$$\mu_{\underset{\sim}{R}}\,(x,z) \geq \underset{y}{MAX}\left\{MIN\left(\mu_{\underset{\sim}{R}}\,(x,y), \mu_{\underset{\sim}{R}}\,(y,z)\right)\right\}.$$

This relation defines the property of transitivity of a fuzzy relation. Such a relation may also be written using the notation

$$\mu_{\underset{\sim}{R}}\,(x,z) \geq \bigvee_{y}\left\{\left(\mu_{\underset{\sim}{R}}\,(x,y) \wedge \mu_{\underset{\sim}{R}}\,(y,z)\right)\right\}$$

where, we recall, $\vee$ is a symbol signifying “maximum of” and $\wedge$ is a symbol signifying “minimum of”.

Example :

$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	0,2	1	0,4	0,4
B	0	0,6	0,3	0
C	0	1	0,3	0
D	0,1	1	1	0,1

(a)

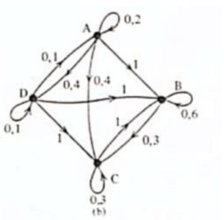


Figure 15.4

Example : Consider the relation $x \underset{\sim}{R} y$, where $x, y \in M, with$

$$\mu_{\underset{\sim}{R}}(x,y) = 0 \quad , \quad y < x,$$
$$= e^{-x} \quad , y \geq x.$$

The matrix of this relation is presented in Figure 15.5. Figure 15.6 shows the results of calculating the right-hand member of

$$\mu_{\underset{\sim}{R}}(x,z) \geq \bigvee_{y} \left\{ \left(\mu_{\underset{\sim}{R}}(x,y) \wedge \mu_{\underset{\sim}{R}}(y,z) \right) \right\}$$

By comparing the two figures we may verify that the above relation is satisfied for all pairs. This relation is transitive.

$\underset{\sim}{R}$ (x \ y)	0	1	2	3	4	5	6	7	...
0	1	1	1	1	1	1	1	1	...
1	0	e^{-1}	e^{-1}	e^{-1}	e^{-1}	e^{-1}	e^{-1}	e^{-1}	...
2	0	0	e^{-2}	e^{-2}	e^{-2}	e^{-2}	e^{-2}	e^{-2}	...
3	0	0	0	e^{-3}	e^{-3}	e^{-3}	e^{-3}	e^{-3}	...
4	0	0	0	0	e^{-4}	e^{-4}	e^{-4}	e^{-4}	...
5	0	0	0	0	0	e^{-5}	e^{-5}	e^{-5}	...
6	0	0	0	0	0	0	e^{-6}	e^{-6}	...
7	0	0	0	0	0	0	0	e^{-7}	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 15.5

$\underset{\sim}{R} \circ \underset{\sim}{R}$ (y)	0	1	2	3	4	5	6	7	...
0	1	1	1	1	1	1	1	1	...
1	0	e^{-1}	e^{-1}	e^{-1}	e^{-1}	e^{-1}	e^{-1}	e^{-1}	...
2	0	0	e^{-2}	e^{-2}	e^{-2}	e^{-2}	e^{-2}	e^{-2}	...
3	0	0	0	e^{-3}	e^{-3}	e^{-3}	e^{-3}	e^{-3}	...
4	0	0	0	0	e^{-4}	e^{-4}	e^{-4}	e^{-4}	...
5	0	0	0	0	0	e^{-5}	e^{-5}	e^{-5}	...
6	0	0	0	0	0	0	e^{-6}	e^{-6}	...
7	0	0	0	0	0	0	0	e^{-7}	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 15.6

16. TRANSITIVE CLOSURE OF A FUZZY BINARY RELATION

Let$\underline{\mathcal{R}}$ be a fuzzy relation in $E \times E$ then define

$$\underline{\mathcal{R}}^2 = \underline{\mathcal{R}} \circ \underline{\mathcal{R}},$$

by

$$\mu_{\underline{\mathcal{R}}}(x,z) = \underset{y}{\text{MAX}} \left[MIN \left(\mu_{\underline{\mathcal{R}}}(x,y), \mu_{\underline{\mathcal{R}}}(y,z) \right) \right]$$

where$x, y, z \in E$ The expression (17.2) may be rewritten

$$\mu_{\underline{\mathcal{R}}^2}(x,z) = \bigvee_{y} \left[\mu_{\underline{\mathcal{R}}}(x,y) \wedge \mu_{\underline{\mathcal{R}}}(y,z) \right]$$

Property (16.8) or (16.9) defining transitivity may also be presented in the following fashion:

$$\underline{\mathcal{R}} \circ \underline{\mathcal{R}} \subset \underline{\mathcal{R}}$$

Suppose

$$\underline{\mathcal{R}}^2 \subset \underline{\mathcal{R}}.$$

and also that

$$\underline{\mathcal{R}}^{k+1} \subset \underline{\mathcal{R}}, k = 1,2,3, \ldots.$$

Then also, evidently,

$$\underline{\mathcal{R}}^k \subset \underline{\mathcal{R}}, k = 1,2,3, \ldots.$$

We shall call the transitive closure of a fuzzy binary relation the relation

$$\hat{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \underline{\mathcal{R}}^3 \cup \ldots.$$

Theorem 1. The transitive closure of any fuzzy binary relation in a transitive binary relation.

Proof. According to (17,8), we may write

$$\hat{\underline{\mathcal{R}}}^2 = \hat{\underline{\mathcal{R}}} \circ \hat{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \underline{\mathcal{R}}^3 \cup \ldots.$$

Then, comparing (17,8) and (17,9), we may write

$$\hat{\underline{\mathcal{R}}}^2 \subset \hat{\underline{\mathcal{R}}}$$

which proves$\underline{\mathcal{R}}$ that is transitive.

To summarize, we have the following properties:

$$\left(\underline{\mathcal{R}} \supset \underline{\mathcal{R}}^2\right) \Leftrightarrow \left(\underline{\mathcal{R}} = \hat{\underline{\mathcal{R}}}\right) \Leftrightarrow \left(\underline{\mathcal{R}}\; is\; transitive\right),$$

$$\left(\underline{\mathcal{R}} = \underline{\mathcal{R}}^2\right) \Leftrightarrow \left(\underline{\mathcal{R}} = \hat{\underline{\mathcal{R}}}\right) \Leftrightarrow \left(\underline{\mathcal{R}}\; is\; transitive\right)$$

Remark. Theorem I gives the means for constructing a transitive relation from any relation. Such a synthesis may have interest.

Theorem II. Let $\underline{\mathcal{R}}$ be any fuzzy hinary relation. If, for some k, one has

$$\underline{\mathcal{R}}^{k+1} = \underline{\mathcal{R}}^k$$

then

$$\hat{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \underline{\mathcal{R}}^3 \cup \ldots \cup \underline{\mathcal{R}}^k.$$

We shall note that the reverse is not true.

Proof. The proof is almost trivial. One has

$$\hat{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \ldots \cup \underline{\mathcal{R}}^k \cup \underline{\mathcal{R}}^{k+1} \cup \underline{\mathcal{R}}^{k+2} \cup \ldots$$

$$= \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \ldots \cup \underline{\mathcal{R}}^k \cup \underline{\mathcal{R}}^k \cup \underline{\mathcal{R}}^k \cup \ldots$$

$$= \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \ldots \cup \underline{\mathcal{R}}^k$$

We shall prove later that, if $\underline{\mathcal{R}} \subset E \times E$, where E is finite and $card(E) = n$, then

$$\hat{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \ldots \cup \underline{\mathcal{R}}^n$$

and there exists a k defined by (17,14) such that $k \leq n$;

We consider several examples,

Example 1. Consider the relation $\underline{\mathcal{R}}$ given in Figure 17.la. One may calculate $\underline{\mathcal{R}}^2$(Figure 17.1b), then $\underline{\mathcal{R}}^3$ (Figure 17.1c). We see that$\underline{\mathcal{R}}^3 = \underline{\mathcal{R}}^2$; one may then stop there, and$\underline{\mathcal{R}}$ is given in Figure 17.1d.

$$\begin{array}{c|c|c|c|}
\underset{\sim}{\mathcal{R}} & A & B & C \\ \hline
A & 0,8 & 1 & 0,1 \\ \hline
B & 0 & 0,4 & 0 \\ \hline
C & 0,3 & 0 & 0,2 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|c|}
\underset{\sim}{\mathcal{R}}^2 & A & B & C \\ \hline
A & 0,8 & 0,8 & 0,1 \\ \hline
B & 0 & 0,4 & 0 \\ \hline
C & 0,3 & 0,3 & 0,2 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|c|}
\underset{\sim}{\mathcal{R}}^3 & A & B & C \\ \hline
A & 0,8 & 0,8 & 0,1 \\ \hline
B & 0 & 0,4 & 0 \\ \hline
C & 0,3 & 0,3 & 0,2 \\ \hline
\end{array}$$

(a) (b) (c)

$$\hat{\underset{\sim}{\mathcal{R}}} = \overset{\underset{\sim}{\mathcal{R}}}{\begin{array}{|c|c|c|} \hline 0,8 & 1 & 0,1 \\ \hline 0 & 0,4 & 0 \\ \hline 0,3 & 0 & 0,2 \\ \hline \end{array}} \cup \overset{\underset{\sim}{\mathcal{R}}^2}{\begin{array}{|c|c|c|} \hline 0,8 & 0,8 & 0,1 \\ \hline 0 & 0,4 & 0 \\ \hline 0,3 & 0,3 & 0,2 \\ \hline \end{array}} = \overset{\hat{\underset{\sim}{\mathcal{R}}}}{\begin{array}{|c|c|c|} \hline 0,8 & 1 & 0,1 \\ \hline 0 & 0,4 & 0 \\ \hline 0,3 & 0,3 & 0,2 \\ \hline \end{array}}$$

(d)

Fig. 16.1

In Figure 17.2 we have verified that

$$\hat{\underline{\mathcal{R}}}^2 \subset \hat{\underline{\mathcal{R}}}$$

$$\hat{\underset{\sim}{\mathcal{R}}}^2 = \overset{\hat{\underset{\sim}{\mathcal{R}}}}{\begin{array}{|c|c|c|} \hline 0,8 & 1 & 0,1 \\ \hline 0 & 0,4 & 0 \\ \hline 0,3 & 0,3 & 0,2 \\ \hline \end{array}} \circ \overset{\hat{\underset{\sim}{\mathcal{R}}}}{\begin{array}{|c|c|c|} \hline 0,8 & 1 & 0,1 \\ \hline 0 & 0,4 & 0 \\ \hline 0,3 & 0,3 & 0,2 \\ \hline \end{array}} = \overset{\hat{\underset{\sim}{\mathcal{R}}}^2}{\begin{array}{|c|c|c|} \hline 0,8 & 0,8 & 0,1 \\ \hline 0 & 0,4 & 0 \\ \hline 0,3 & 0,3 & 0,2 \\ \hline \end{array}}.$$

Fig. 16.2

Example 2. In the example presented in Figure 17.3, we have a relation$\underset{\sim}{\mathcal{R}}$ that is transitive, By carrying out the calculations in the same order as the above, one sees that

$$\hat{\underset{\sim}{\mathcal{R}}} = \underset{\sim}{\mathcal{R}}$$

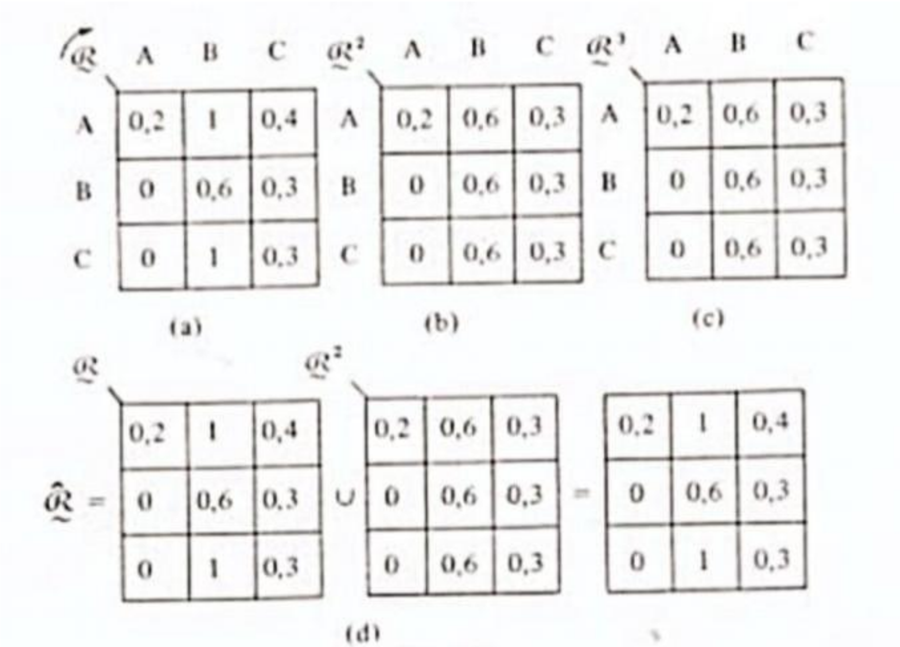
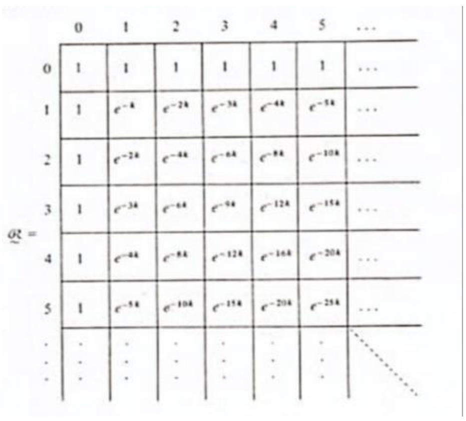


Fig. 16.3

Example 3. Consider the relation $x\underset{\sim}{\mathcal{R}}x$, where $x, y \in\ N$ with

$$\mu_{\underset{\sim}{\mathcal{R}}}(x,y) = e^{-kxy},$$

with$k > 1$ sufficiently large so that this membership function expresses the relation "x and y are both rather small nonnegative integers."† For a matrix representation of this relation, one has



†One may say for this that among the two elements of the ordered pair (x,y) there is at leas that is rather small,

Calculation of $\underline{\mathcal{R}}^2$ gives

$$\underline{\mathcal{R}}^2 = \begin{array}{c|c|c|c|c|c|c|c}
 & 0 & 1 & 2 & 3 & 4 & 5 & \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
2 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
3 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
4 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
5 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Thus, since$\underline{\mathcal{R}}^2 \supset \underline{\mathcal{R}}$ instead of $\underline{\mathcal{R}}^2 \subset \underline{\mathcal{R}}$, this fuzzy relation is not transitive.

A similar and easy proof would show that it is the same if $x, x \in R^+$ instead of N.

We shall return in Section 29, as promised in Section 15, to the case where E is not finite.

Example 4. We return to the case of a relation $\underline{\mathcal{R}} \subset E \times E$, where E is finite. This is done in order to make it clear that one does not always have the favorable case (17.13). But we shall go on to show also from this example a very interesting phenomenon.

In Figure 17.4 we have given a relation$\underline{\mathcal{R}}$ and successively calculated $\underline{\mathcal{R}}^2, \underline{\mathcal{R}}^3, \ldots$. One notices that this does not converge; there does not exist a fixed k after which $\underline{\mathcal{R}}^{k+1} = \underline{\mathcal{R}}^k$.

Fortunately, thanks to (17.16) we know that we may stop at $k = 3$. And then one obtains $\underline{\mathcal{R}}$ easily.

But, if the reader considers attentively all the relations obtained, he sees that for $k > 3$, we have

$$\underline{\mathcal{R}}^4 = \underline{\mathcal{R}}^6 = \cdots = \underline{\mathcal{R}}^{2r} = \underline{\mathcal{R}}^{2r+2} = \cdots = \underline{\mathcal{R}}_p$$
$$\underline{\mathcal{R}}^5 = \underline{\mathcal{R}}^7 = \cdots = \underline{\mathcal{R}}^{2r+1} = \underline{\mathcal{R}}^{2r+3} = \cdots = \underline{\mathcal{R}}_j$$

Thus there appears a cyclic phenomenon that would be interesting to study. We lack room here to study "cyclic fuzzy relations," which we leave with these remarks, but we commend these to the reader who perhaps finds himself interested.

$\mathcal{R}$	A	B	C
A	0,3	0,6	0,2
B	1	0,2	0,7
C	0	0,8	0,1

$\mathcal{R}^2$	A	B	C
A	0,6	0,3	0,6
B	0,3	0,7	0,2
C	0,8	0,2	0,7

$\mathcal{R}^3$	A	B	C
A	0,3	0,6	0,3
B	0,7	0,3	0,7
C	0,3	0,7	0,2

$\mathcal{R}^4$	A	B	C
A	0,6	0,3	0,6
B	0,3	0,7	0,3
C	0,7	0,3	0,7

$\mathcal{R}^5$	A	B	C
A	0,3	0,6	0,3
B	0,7	0,3	0,7
C	0,3	0,7	0,3

$\mathcal{R}^6$	A	B	C
A	0,6	0,3	0,6
B	0,3	0,7	0,3
C	0,7	0,3	0,7

$\mathcal{R}^7$	A	B	C
A	0,3	0,6	0,3
B	0,7	0,3	0,7
C	0,3	0,7	0,3

$\mathcal{R}^8$	A	B	C
A	0,6	0,3	0,6
B	0,3	0,7	0,3
C	0,7	0,3	0,7

$\mathcal{R}^9$	A	B	C
A	0,3	0,6	0,3
B	0,7	0,3	0,7
C	0,3	0,7	0,3

$\mathcal{R}^{10}$	A	B	C
A	0,6	0,3	0,6
B	0,3	0,7	0,3
C	0,7	0,3	0,7

. . . .

$\mathcal{R}$	A	B	C
A	0,6	0,6	0,6
B	1	0,7	0,7
C	0,8	0,8	0,7

Fig. 16.4

Remark. One may ask the following interesting question: Does the composition of two transitive relations $\underline{\mathcal{R}}_1$ and $\underline{\mathcal{R}}_2$ always give a relation $\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2$ and/or$\underline{\mathcal{R}}_2 \circ \underline{\mathcal{R}}_1$ that is transitive? This is, unfortunately, not the case, as the following counterexample shows:

Example. Let $\underline{\mathcal{R}}_1$ be as given in (17.24); by checking the property $\underline{\mathcal{R}}_1^2 \subset \underline{\mathcal{R}}_1$ one may verify that this relation is indeed transitive:

$\mathcal{R}_1$	A	B	C	D	E
A	0,5	0,9	0	0	0,5
B	0	0,7	0	0	0
C	0	1	0,1	0,1	0
D	0	1	0,4	1	0
E	0,7	0,9	0	0	0,5

$\circ$

$\mathcal{R}_1$	A	B	C	D	E
A	0,5	0,9	0	0	0,5
B	0	0,7	0	0	0
C	0	1	0,1	0,1	0
D	0	1	0,4	1	0
E	0,7	0,9	0	0	0,5

$=$

$\mathcal{R}_1^2$	A	B	C	D	E
A	0,5	0,7	0	0	0,5
B	0	0,7	0	0	0
C	0	0,7	0,1	0,1	0
D	0	1	0,4	1	0
E	0,5	0,7	0	0	0,5

Let $\underline{\mathcal{R}}_2$, be as in (17.25), by checking the property $\underline{\mathcal{R}}_2^2 \subset \underline{\mathcal{R}}_2$,one verifies that this relation also is transitive:

$$
\begin{array}{c|ccccc}
\mathcal{R}_1 & A & B & C & D & E \\
\hline
A & 0{,}7 & 0 & 0 & 0 & 0 \\
B & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
C & 0 & 0 & 0{,}5 & 0{,}5 & 0 \\
D & 0 & 0 & 0{,}2 & 0{,}4 & 0 \\
E & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1
\end{array}
\circ
\begin{array}{c|ccccc}
\mathcal{R}_2 & A & B & C & D & E \\
\hline
A & 0{,}7 & 0 & 0 & 0 & 0 \\
B & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
C & 0 & 0 & 0{,}5 & 0{,}5 & 0 \\
D & 0 & 0 & 0{,}2 & 0{,}4 & 0 \\
E & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1
\end{array}
=
\begin{array}{c|ccccc}
\mathcal{R}_2^2 & A & B & C & D & E \\
\hline
A & 0{,}7 & 0 & 0 & 0 & 0 \\
B & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
C & 0 & 0 & 0{,}5 & 0{,}5 & 0 \\
D & 0 & 0 & 0{,}2 & 0{,}4 & 0 \\
E & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1
\end{array}
$$

We now calculate $\underline{\mathcal{R}}_2 \subset \underline{\mathcal{R}}_1$:

$$
\begin{array}{c|ccccc}
\mathcal{R}_1 & A & B & C & D & E \\
\hline
A & 0{,}5 & 0{,}9 & 0 & 0 & 0{,}5 \\
B & 0 & 0{,}7 & 0 & 0 & 0 \\
C & 0 & 1 & 0{,}1 & 0{,}1 & 0 \\
D & 0 & 1 & 0{,}4 & 1 & 0 \\
E & 0{,}7 & 0{,}9 & 0 & 0 & 0{,}5
\end{array}
\circ
\begin{array}{c|ccccc}
\mathcal{R}_2 & A & B & C & D & E \\
\hline
A & 0{,}7 & 0 & 0 & 0 & 0 \\
B & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
C & 0 & 0 & 0{,}5 & 0{,}5 & 0 \\
D & 0 & 0 & 0{,}2 & 0{,}4 & 0 \\
E & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1
\end{array}
=
\begin{array}{c|ccccc}
\mathcal{R}_2 \circ \mathcal{R}_1 & A & B & C & D & E \\
\hline
A & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9 \\
B & 0{,}7 & 0{,}7 & 0{,}6 & 0{,}6 & 0{,}7 \\
C & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
D & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
E & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9
\end{array}
$$

And $\left(\underline{\mathcal{R}}_2 \circ \underline{\mathcal{R}}_1\right)^2$

$$
\begin{array}{c|ccccc}
\mathcal{R}_2 \circ \mathcal{R}_1 & A & B & C & D & E \\
\hline
A & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9 \\
B & 0{,}7 & 0{,}7 & 0{,}6 & 0{,}6 & 0{,}7 \\
C & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
D & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
E & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9
\end{array}
\circ
\begin{array}{c|ccccc}
\mathcal{R}_2 \circ \mathcal{R}_1 & A & B & C & D & E \\
\hline
A & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9 \\
B & 0{,}7 & 0{,}7 & 0{,}6 & 0{,}6 & 0{,}7 \\
C & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
D & 0{,}8 & 1 & 0{,}6 & 0{,}6 & 1 \\
E & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9
\end{array}
=
\begin{array}{c|ccccc}
(\mathcal{R}_2 \circ \mathcal{R}_1)^2 & A & B & C & D & E \\
\hline
A & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9 \\
B & 0{,}7 & 0{,}7 & 0{,}6 & 0{,}6 & 0{,}7 \\
C & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9 \\
D & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9 \\
E & 0{,}8 & 0{,}9 & 0{,}6 & 0{,}6 & 0{,}9
\end{array}
\quad (17.27)
$$

The relation $\left(\underline{\mathcal{R}}_2 \circ \underline{\mathcal{R}}_1\right)^2 \subset \underline{\mathcal{R}}_2 \circ \underline{\mathcal{R}}_1$ is certainly verified.

We now calculate $\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2$

$\underline{\mathcal{R}}_1$	A	B	C	D	E
A	0,7	0	0	0	0
B	0,8	1	0,6	0,6	1
C	0	0	0,5	0,5	0
D	0	0	0,2	0,4	0
E	0,8	1	0,6	0,6	1

$\circ$

$\underline{\mathcal{R}}_2$	A	B	C	D	E
A	0,5	0,9	0	0	0,5
B	0	0,7	0	0	0
C	0	1	0,1	0,1	0
D	0	1	0,4	1	0
E	0,7	0,9	0	0	0,5

$=$

$\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2$	A	B	C	D	E
A	0,5	0,7	0	0	0,5
B	0,7	0,9	0,4	0,6	0,5
C	0	0,5	0,4	0,5	0
D	0	0,4	0,4	0,4	0
E	0,7	0,9	0,4	0,6	0,5

And $\left(\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2\right)^2$:

(29)

$\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2$	A	B	C	D	E
A	0,5	0,7	0	0	0,5
B	0,7	0,9	0,4	0,6	0,5
C	0	0,5	0,4	0,5	0
D	0	0,4	0,4	0,4	0
E	0,7	0,9	0,4	0,6	0,5

$\circ$

$\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2$	A	B	C	D	E
A	0,5	0,7	0	0	0,5
B	0,7	0,9	0,4	0,6	0,5
C	0	0,5	0,4	0,5	0
D	0	0,4	0,4	0,4	0
E	0,7	0,9	0,4	0,6	0,5

$=$

$(\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2)^2$	A	B	C	D	E
A	0,7	0,7	0,4	0,6	0,5
B	0,7	0,9	0,4	0,6	0,5
C	0,5	0,5	0,4	0,5	0,5
D	0,4	0,4	0,4	0,4	0,4
E	0,7	0,9	0,4	0,6	0,5

One sees that we do not have$\left(\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2\right)^2 \subset \underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2$, and it follows that $\underline{\mathcal{R}}_1 \circ \underline{\mathcal{R}}_2$ not transitive.

Thus, the composition of two tramitive relations will not always give a transitive relation.

17. A PATH IN A FINITE FUZZY GRAPH

We shall consider in the finite graph $G \subset E \times E$ an ordered tuple with or without repetition†.

$$C = \left(x_{i_1}, x_{i_2}, \dots, x_{i_r}\right)$$

where the $x_{i_k} \in E, k = 1,2, \dots, r$ and with the condition

$$\forall\left(x_{i_k}, x_{i_{k+1}}\right): \mu_{\underline{\mathcal{R}}}\left(x_{i_k}, x_{i_{k+1}}\right) > 0, k = 1,2, \dots, r-1$$

Such an ordered r-tuple will be called a path from x_{i_1} to x_{i_r} in the graph$G$ (one also says in the relation$\underline{\mathcal{R}}$

With each path $\left(x_{i_1}, x_{i_2}, \dots, x_{i_r}\right)$ we shall associate a value defined by

$$l\left(x_{i_1}, x_{i_2}, \dots, x_{i_r}\right) = \mu_{\underline{\mathcal{R}}}\left(x_{i_1}, x_{i_2}\right) \wedge \mu_{\underline{\mathcal{R}}}\left(x_{i_2}, x_{i_3}\right) \wedge \dots \wedge \mu_{\underline{\mathcal{R}}}\left(x_{i_{r-1}}, x_{i_r}\right)$$

We now consider all possible paths existing between x_i and x_j, two arbitrary elements of E; let $C(x_i, x_j)$ be the ordinary set of all such paths:

$$C(x_i, x_j) = \{c(x_i, x_j) c(x_i, x_j) = (x_{i_1} = x_i, x_{i_2}, x_{i_{r-1}}, x_{i_r} = x_j)\}$$

We shall define the strongest path $C^*(x_i, x_j)$ from x_i to x_j by

$$l^*(x_i, x_j) = \bigvee_{C(x_i, x_j)} l(x_{i_1} = x_i, x_{i_2}, \dots, x_{i_{r-1}}, x_{i_r} = x_j)$$

Also, the number of elements less one constituting a path will be called the length of the path.

Before giving several examples, we consider various theorems.

Theorem I. Let $\underline{\mathcal{R}} \subset E \times E$; then one has

$$\forall (x, y) \in E \times E: \mu_{\underline{\mathcal{R}}^k}(x, y) = l_k^*(x, y),$$

where $l_k^*(x, y)$ is the strongest path existing from x to y of length k.

Proof. The result is immediate, it suffices to consider (18,4) and (18.3) on the one hand, and from there the composition $\underline{\mathcal{R}} \circ \underline{\mathcal{R}} \circ \dots \circ \underline{\mathcal{R}}$. It is in fact the same max-min operation presented in two different fashions.

†In other words, may be less than, equal to, or greater than card E. depending on the case.

Theorem II. Let $\underline{\mathcal{R}} \subset E \times E$ and $\underline{\hat{\mathcal{R}}}$ be the transitive closure of $\underline{\mathcal{R}}$; then one has

$$\forall (x, y) \in E \times E: \mu_{\underline{\hat{\mathcal{R}}}}(x, y) = l^*(x, y)$$

Proof. It suffices to review the definitions of $\underline{\hat{\mathcal{R}}}$ and of $l^*(x, y)$.

Theorem III. Let $n = card\ E$; if k is the length of a path from x_i to x_j, with $k > n = card\ E$, then all the elements of the chain are not unique, there is at least one "circuit" (closed path) in the path. If one removes this for these) circuit(s), the resulting path has a length less than or equal to n; one may also state

$$l_k^*(x, y) = l_{i \le n}^*(x, y)$$

where $l_{i \le n}^*(x, y)$ is the value of the strongest path of length less than or equal to n from x to y.

Proof. After removing the circuits there remains a chain that has at most length n; relation (15,7) is then verified,

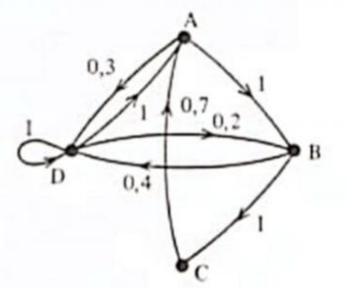
Theorem IV.† If $\underset{\sim}{\mathcal{R}} \subset E \times E$and $n = card\ E$, then

$$\hat{\underset{\sim}{\mathcal{R}}} = \underset{\sim}{\mathcal{R}} \cup \underset{\sim}{\mathcal{R}}^2 \cup \ldots \cup \underset{\sim}{\mathcal{R}}^n$$

Proof. This follows immediately from Theorem II [see (18,6)].

Example. We consider the relation $\underset{\sim}{\mathcal{R}}$ represented in Figure 18.1. The results presented in Figure 17,2 will be used in our explanations, Let (B, C, A, D) be a path. We calculate its value:

$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	0	1	0	0,3
B	0	0	1	0,4
C	0,7	0	0	0
D	1	0,2	0	1



(a) (b)

(the ordered pairs (x, y) such that$\mu_{\underset{\sim}{\mathcal{R}}}(x, y) = 0$ have not been represented)

Fig. 17.1

$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	0	1	0	0,3
B	0	0	1	0,4
C	0,7	0	0	0
D	1	0,2	0	1

(a) $\underset{\sim}{\mathcal{R}}$

0,3	0,2	1	0,4
0,7	0,2	0	0,4
0	0,7	0	0,3
1	1	0,2	1

(b) $\underset{\sim}{\mathcal{R}}^2$

0,3	1	1	0,4
0,7	0,2	1	0,4
0,7	0,7	0	0,3
1	1	0,2	1

(c) $\underset{\sim}{\mathcal{R}} \cup \underset{\sim}{\mathcal{R}}^2$

0,7	0,3	0,2	0,4
0,4	0,7	0,2	0,4
0,3	0,2	0,7	0,4
1	1	1	1

(d) $\underset{\sim}{\mathcal{R}}^3$

0,7	1	1	0,4
0,7	0,7	1	0,4
0,7	0,7	0,7	0,4
1	1	1	1

(e) $\underset{\sim}{\mathcal{R}} \cup \underset{\sim}{\mathcal{R}}^2 \cup \underset{\sim}{\mathcal{R}}^3$

0,4	0,7	0,3	0,4
0,4	0,4	0,7	0,4
0,7	0,3	0,2	0,4
1	1	1	1

(f) $\underset{\sim}{\mathcal{R}}^4$

0,7	1	1	0,4
0,7	0,7	1	0,4
0,7	0,7	0,7	0,4
1	1	1	1

$\underset{\sim}{\mathcal{R}} \cup \underset{\sim}{\mathcal{R}}^2 \cup \underset{\sim}{\mathcal{R}}^3 \cup \underset{\sim}{\mathcal{R}}^4 = \hat{\underset{\sim}{\mathcal{R}}}$

(g)

Fig. 17.2

$$l(B,C,A,D) = \mu_{\underline{\mathcal{R}}}(B,C) \wedge \mu_{\underline{\mathcal{R}}}(C,A) \wedge \mu_{\underline{\mathcal{R}}}(A,D)$$

$$= 1 \wedge 0{,}7 \wedge 0{,}3 = 0{,}3.$$

Now we examine all paths† from B to D whose length is less than or equal to 3; these are only the paths $(B,D),(B,D,D),(B,D,D,D)$, for which we have

$$l(B,D) = \mu_{\underline{\mathcal{R}}}(B,D) = 0{,}4, L(B,D,D) = \mu_{\underline{\mathcal{R}}}(B,D) \wedge \mu_{\underline{\mathcal{R}}}(D,D) = 0{,}4 \wedge 1 = 0{,}4$$

†One knows how to carry out such an enumeration in an automatic fashion, without omission and without repetition, See, for example, the references [1K.2K]

$$l(B,D,D,D) = \mu_{\underline{\mathcal{R}}}(B,D) \wedge \mu_{\underline{\mathcal{R}}}(D,D) \wedge \mu_{\underline{\mathcal{R}}}(D,D) = 0{,}4 \wedge 1 \wedge 1 = 0{,}4$$

One then has

$$l^*(B,D) = l(B,C,A,D) \vee l(B,D) \vee l(B,D,D) \vee l(B,D,D,D)$$

$$= 0{,}3 \vee 0{,}4 \vee 0{,}4 \vee 0{,}4 = 0{,}4$$

If we locate $\underline{\mathcal{R}}$ in Figure 17.2g, we find

$$\mu_{\underline{\hat{\mathcal{R}}}}(B,D) = 0{,}4\ (Theorem\ II - 17.6)$$

On the other hand, there are two paths of length 3 between $B$and D; these are(B,C,A,D) and$(B\ D,D,D)$ One then has

$$l^*(B,D) = l(B,C,A,D) \vee l(B,D,D,D)$$

$$= 0{,}3 \vee 0{,}4 = 0{,}4$$

One verifies with†

$$\mu_{\underline{\mathcal{R}}^3}(B,D) = 0{,}4\ (Theorem\ I - 1.5)$$

Now consider the path (C,A,B,D,A,D). This path possesses a circuit (D,A,D) eliminating this we see

$$l_5^*(C,D) = l_{i\leq 4}^*(C,D)$$

$$= l_1^*(C,D) \vee l_2^*(C,D) \vee l_3^*(C,D) \vee l_4^*(C,D)$$

$$= \mu_{\underline{\mathcal{R}}}(C,D) \vee \mu_{\underline{\mathcal{R}}^2}(C,D) \vee \mu_{\underline{\mathcal{R}}^3}(C,D) \vee \mu_{\underline{\mathcal{R}}^4}(C,D)$$

$$= 0 \vee 0{,}3 \vee 0{,}4 \vee 0{,}4 = 0{,}4$$

One may have expected to find 0,3; but the strongest path of length 5 between $C$and D is not (C,A,B,D,A,D) but (C,A,B,D,D,D), these two, moreover, reduce to (C,A,B,D) when the circuits are eliminated. All this may be seen clearly in Figure 18.1b.

Notion of a path defined with respect to other operators.Max-star transitivity. The value defined with the aid of expression (18.3) may be extended, in its definition, to operators other than $\wedge$, under the restriction that those considered have the properties of associativity and monotonicity. If$*$ is such an operator, one then sees

$$l(x_{i_1}, x_{i_2}, \dots, x_{i_r}) = \mu_{\underline{\mathcal{R}}}(x_{i_1}, x_{i_2}) * \mu_{\underline{\mathcal{R}}}(x_{i_2}, x_{i_3}) * \dots * \mu_{\underline{\mathcal{R}}}(x_{i_{r-1}}, x_{i_r})$$

In particular, if is the product operator, denoted and defined by (12.35), one sees

$$l(x_{i_1}, x_{i_2}, \dots, x_{i_r}) = \mu_{\underline{\mathcal{R}}}(x_{i_1}, x_{i_2}). \mu_{\underline{\mathcal{R}}}(x_{i_2}, x_{i_3}). \dots . \mu_{\underline{\mathcal{R}}}(x_{i_{r-1}}, x_{i_r})$$

Due to the property

$$a, b \leq a \wedge b \ if \ a, b \in [0,1].$$

†One finds here that $l_3^*(B, D) = l^*(B, D)$, but this is fortuitious-the relations $\underline{\mathcal{R}}^3$ and $\hat{\underline{\mathcal{R}}}$ being different (see Figures 15.2d and 17.2g).

UNIT III

FUZZY RELATIONS

18. RELATION OF FUZZY PREORDER

A binary fuzzy relation that is

(1) Transitive [(16.9)]

(2) Reflexive [(16.7)]

is a relation of fuzzy preorder.

First we consider an important theorem.

Theorem I. If $\underline{\mathcal{R}}$ is transitive and reflexive (that is, is a preorder), then

(18.1) $$\underline{\mathcal{R}}^2 = \underline{\mathcal{R}}, k = 1,2,3, \dots$$

Proof. It suffices to review the definition of transitivity [(16.9) and (17.5)] and to show

(18.2) $$\underline{\mathcal{R}}^2 = \underline{\mathcal{R}},$$

If one asserts that

(18.3) $$\forall x: \mu_{\underline{\mathcal{R}}}(x, x) = 1$$

Since

$$\underline{\mathcal{R}}^2 = \underline{\mathcal{R}}^2 \circ \underline{\mathcal{R}},$$

One has, according to (13.2)

(18.4) $$\mu_{\underline{\mathcal{R}}^2}(x, z) = \bigvee_y [\mu_{\underline{\mathcal{R}}}(x, x) \wedge \mu_{\underline{\mathcal{R}}}(y, z)]$$

The right-hand member of (18.4) contains two equal terms

(18.5) $$\mu_{\underline{\mathcal{R}}}(x,x) \wedge \mu_{\underline{\mathcal{R}}}(x,z) = \mu_{\underline{\mathcal{R}}}(x,z) \wedge \mu(z,z) = \mu_{\underline{\mathcal{R}}}(x,z)$$

because

(18.6) $$\mu_{\underline{\mathcal{R}}}(x,x) = \mu_{\underline{\mathcal{R}}}(z,z) = 1 \qquad \text{reflexivity}$$

Recall that $\underline{\mathcal{R}}$ is transitive (16.9), that is,

(18.7) $$\mu_{\underline{\mathcal{R}}}(x,z) \geq \bigvee_{y} [\mu_{\underline{\mathcal{R}}}(x,y) \wedge \mu_{\underline{\mathcal{R}}}(y,z)]$$

it then results that$\mu_{\underline{\mathcal{R}}}(x,y)$ is greater than or equal to the terms $\mu_{\underline{\mathcal{R}}}(x,y) \wedge \mu_{\underline{\mathcal{R}}}(y,z)$ thisis then the value of the member on the right of (19.4), and one indeed has

(18.8) $$\underline{\mathcal{R}}^2 = \underline{\mathcal{R}}, \qquad Q.E.D$$

Theorem II. If $\underline{\mathcal{R}}$ is a preorder, then

(18.9) $$\underline{\mathcal{R}} = \underline{\mathcal{R}}^2 = \cdots = \underline{\mathcal{R}}^k = \cdots = \underline{\tilde{\mathcal{R}}}$$

Proof. This is a corollary to Theorem I. It suffices to consider (16.8) and (18.8)together.

Example 1. Figure 18.1 represents a preorder on

(18.10) $$E = \{A, B, C, D, E\}$$

One may verify transitivity with the aid of the relation

(18.11) $$\underline{\mathcal{R}}^2 \subset \underline{\mathcal{R}}$$

Reflexivity is directly apparent from the presence of the ones on the principal diagonal.

Finally, one may verify that one indeed has

(18.12) $$\underline{\mathcal{R}}^2 = \underline{\mathcal{R}}$$

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	1	0,7	0,8	0,5	0,5
B	0	1	0,3	0	0,2
C	0	0,7	1	0	0,2
D	0,6	1	0,9	1	0,6
E	0	0	0	0	1

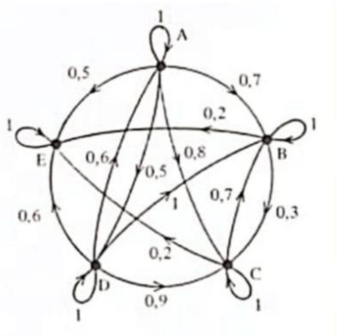


Fig. 18.1

Example 2. Consider a graph $G \subset E \times E.$ where E is finite, and suppose that G is reflexive, Then the binary fuzzy telation "there exists a path from x to yin G" (in the sense of the word *path* given in Section 18) is a preorder.

Example 3. The fuzzy binary relation $x\underline{\mathcal{R}}y$ where $x, y \in N$ with

(18.13) $$\mu_{\underline{\mathcal{R}}}(x, y) = e^{-k(x-y)^2}, with\ k > 1$$

is not a preorder because it is not transitive [see (16.12)].

Example 4. (Figure 18.2).

$\underset{\sim}{\mathcal{R}}$	x_1	x_2	x_3	x_4	x_5	x_6	...
x_1	1	a_1	a_1	a_1	a_1	a_1	...
x_2	0	1	a_2	a_2	a_2	a_2	...
x_3	0	a_1	1	a_3	a_3	a_3	...
x_4	0	a_1	a_2	1	a_4	a_4	...
x_5	0	a_1	a_2	a_3	1	a_5	...
x_6	0	a_1	a_2	a_3	a_4	1	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 18.2

(19.14) $0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots \leq 1$this relation on a denumberably infinite set E is a preorder.

Fuzzy *semipreorder*. A transitive fuzzy relation that is not reflexive is called a *semipreorder*, or what is the same thing, a *nonreflexive fuzzy preorder*,

Example 1. The relation represented in Figure 18.3 is transitive but not reflexive; it is a semipreorder.

$\underset{\sim}{\mathcal{R}}$	A	B	C
A	0,2	1	0,4
B	0	0,6	0,3
C	0	1	0,3

Fig 18.3

Example 2. The relation presented in Figure 16.7 is a semipreorder.

Antireflexive fuzzy preorder. A particular case of a fuzzy semipreorder is that where

(18.15) $$\forall x \in E\colon \mu_{\underset{\sim}{\mathcal{R}}}(x, x) = 0$$

One says then that the fuzzy preorder is antireflexive,

Thus, the preorder relation presented in Figure 18.4 is antireflexive,

$\underset{\sim}{\mathcal{R}}$	A	B	C
A	0	1	0,4
B	0	0	0
C	0	1	0

Fig. 18.4

19. RELATION OF SIMILITUDE

A fuzzy binary relation that is

(1) transitive (16.9)

(2) reflexive (16.7)

(3) symmetric (16.6)

is called a relation of similitude or a fuzzy equivalence relation. It is evidently a preorder.

First, we give several examples,

Example 1. An example is presented in **Figure 20.1.** One may verify reflexivity and symmetry directly. In order to verify transitivity, it suffices to calculate $\underset{\sim}{\mathcal{R}}^2$. One must then have, according to (19.9), (19.1) $$\underline{\mathcal{R}}^2 = \underline{\mathcal{R}}$$

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	1	0,8	0,7	1	0,9
B	0,8	1	0,7	0,8	0,8
C	0,7	0,7	1	0,7	0,7
D	1	0,8	0,7	1	0,9
E	0,9	0,8	0,7	0,9	1

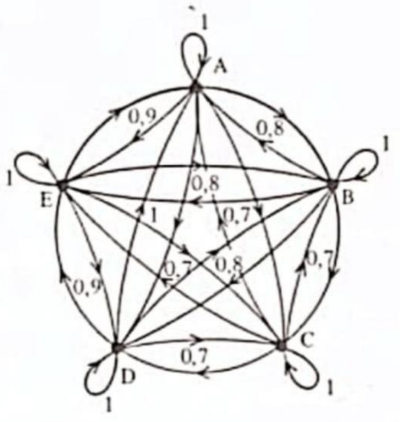


Fig. 19.1

Example 2. (Figare 19.2). If one takes $0 \leq a \leq 1$, then one has a similitude relation.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	1	a	a	a	a
B	a	1	a	a	a
C	a	a	1	a	a
D	a	a	a	1	a
E	a	a	a	a	1

Fig. 20.2

Example 3. (Figure 19.3). If we suppose

(19.2) $$0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots \leq 1$$

this is a similitude relation in an infinite set E.

$\underset{\sim}{\mathcal{R}}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$
x_1	1	a_1	a_1	a_1	a_1	a_1	a_1	$\cdots$
x_2	a_1	1	a_2	a_2	a_2	a_2	a_2	$\cdots$
x_3	a_1	a_2	1	a_3	a_3	a_3	a_3	$\cdots$
x_4	a_1	a_2	a_3	1	a_4	a_4	a_4	$\cdots$
x_5	a_1	a_2	a_3	a_4	1	a_5	a_5	$\cdots$
x_6	a_1	a_2	a_3	a_4	a_5	1	a_6	$\cdots$
x_7	a_1	a_2	a_3	a_4	a_5	a_6	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 19.3

Example 4. The fuzzy relations $x\underline{\mathcal{R}}y$ where $x, y \in R^+$ with

(19.3)
$$\mu_{\underline{\mathcal{R}}}(x,y) \begin{array}{ll} = e^{-k(y+1)} & y < x, k > 1 \\ = 1 & y = x \\ = e^{-k(x+1)} & y > x, k > 1 \end{array}$$

is a similitude relation, as the reader is asked to verify in the Exercises (see also Section 29 a little later).

Theorem 1. Let $\underline{\mathcal{R}} \subset E \times E$be a similitude relation. Let x, y, z be three elements of E. Put

(19.4)
$$\begin{aligned} c &= \mu_{\underline{\mathcal{R}}}(x,z) = \mu_{\underline{\mathcal{R}}}(z,x), \\ a &= \mu_{\underline{\mathcal{R}}}(x,y) = \mu_{\underline{\mathcal{R}}}(y,x), \\ b &= \mu_{\underline{\mathcal{R}}}(y,z) = \mu_{\underline{\mathcal{R}}}(z,y), \end{aligned}$$

Then

(19.5)
$$c \geq a = b \quad or \quad a \geq b = c \quad or \quad b \geq c = a$$

In other words, of these three quantities a, b, and c at least two are equal and the third is greater than the other two.

Proof. One already has by hypothesis

(19.6) $$c \geq a \wedge b,$$

(19.7) $$b \geq c \wedge a,$$

(19.8) $$a \geq b \wedge c.$$

We suppose that we have

(19.9) $$c \geq b > a.$$

then (20.6) and (20.7) are verified, but (20.8) is not, and if one takes $b = a$. the three relations are verified.

Suppose that we have

(19.10) $$c \geq a > b,$$

then (19,6) and (19.8) are verified, but (19.8) is not, and if one takes $a = b$ the three relations are verified,

One then may not have (19,9) nor (19.10), but on the contrary

(19.11) $$c \geq a = b \quad holds.$$

One could show in the same manner that one may not have$a \geq b > c$ or $a \geq c > b$.

But

(19.12) $$a \geq b = c \quad holds.$$

One could show again in the same manner that one may not have $b \geq c > a$ or $b \geq a > c$, but

(19.13) $$b \geq a = c \quad holds.$$

Thus it is necessary that one always have at least two of the values equal.

The inequalities (19.6) - (19.8) then give:

If $a = b$

(19.14) $$\begin{aligned} c &\geq a \wedge b, \\ b &= c \wedge a \\ a &= b \wedge c \end{aligned}$$

If $b = c$

(19.15) $$\begin{aligned} c &= a \wedge b, \\ b &= c \wedge a \\ a &\geq b \wedge c \end{aligned}$$

If $c = a$

(19.16) $$\begin{aligned} c &= a \wedge b, \\ b &\geq c \wedge a \\ a &= b \wedge c \end{aligned}$$

20. SUBRELATION OF SIMILITUDE IN A FUZZY PREORDER

Let $\underline{\mathcal{R}} \subset E \times E$ be a relation of fuzzy preorder. If there exists an ordinary subset $E_1 \subset E$ such that $\forall x, y \in E_1 : \mu_{\underline{\mathcal{R}}}(x, y) = \mu_{\underline{\mathcal{R}}}(y, x)$, the elements of E_1, form among themselves a similitude relation that we shall call a similitude subrelation in the preorder $\underline{\mathcal{R}}$.

We shall say that a similitude subrelation is maximal if there is no other similitude relation of the same nature in the relation being considered.

Suppose now that a preorder relation is such that each of the elements of the reference set involved belongs to a maximal similitude subrelation and does not belong to any other. This may be rephrased: all the maximal subrelations are disjoint. In this case we call the subsets for which one has such disjoint maximal similitude subrelations *similitude classes* of the preorder.

Thus, not all fuzzy preorders are decomposable into similitude classes. We shall consider several examples.

Example 1. The relation shown inFigure 20.1 is certainly a preorder [one may verify this with reference to (18.21)]. A 1 But this preorder is not a symmetric relation. However, note that the relation $\underline{\mathcal{R}}$ may be decomposed into three sub- relations: $\underline{\mathcal{R}}_1$ relative to $\{A, B, C, E, F\}$,$\underline{\mathcal{R}}_2$relative to$\{D\}$, $\underline{\mathcal{R}}_3$ relative to G. The ordinary subsets $K_1 = \{A, B, C, D, E\}, K_2 = \{D\}$, $K_3 = \{G\}$ are clearly maximal for the property of similitude [this is not the case, for example, for $\{B, C, F\}$ or $\{A, C, E\}$]. We shall say that the relation of fuzzy preorder is decomposable into maximal disjoint similitude subrelations relative K_1, K_2 and K_3. forming the similitude classes existing in the preordered set.

$\underset{\sim}{\mathcal{R}}$	A	B	C	E	F	D	G
A	1	0,2	0,2	0,2	0,2	0,3	0,4
B	0,2	1	0,5	0,2	0,2	0,3	0,5
C	0,2	0,5	1	0,2	0,2	0,3	0,5
E	0,2	0,2	0,2	1	0,8	0,3	0,5
F	0,2	0,2	0,2	0,8	1	0,3	0,5
D	0,2	0,2	0,2	0,2	0,2	1	0,4
G	0,2	0,2	0,2	0,2	0,2	0,2	1

Fig. 20.1

If we now consider the strongest paths existing between these classes [see the definition, (17.4)], these classes then form among themselves (Figure 20.2) a transitive

$\underset{\sim}{\mathcal{R}}'$	K_1	K_2	K_3
K_1	1	0,3	0,5
K_2	0,2	1	0,4
K_3	0,2	0,2	1

(a)

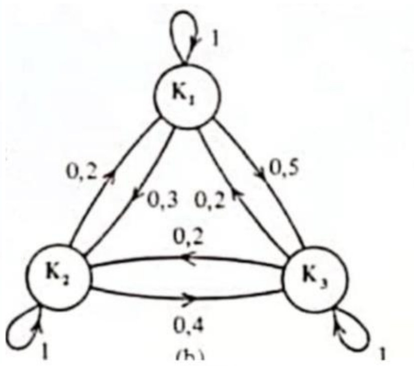
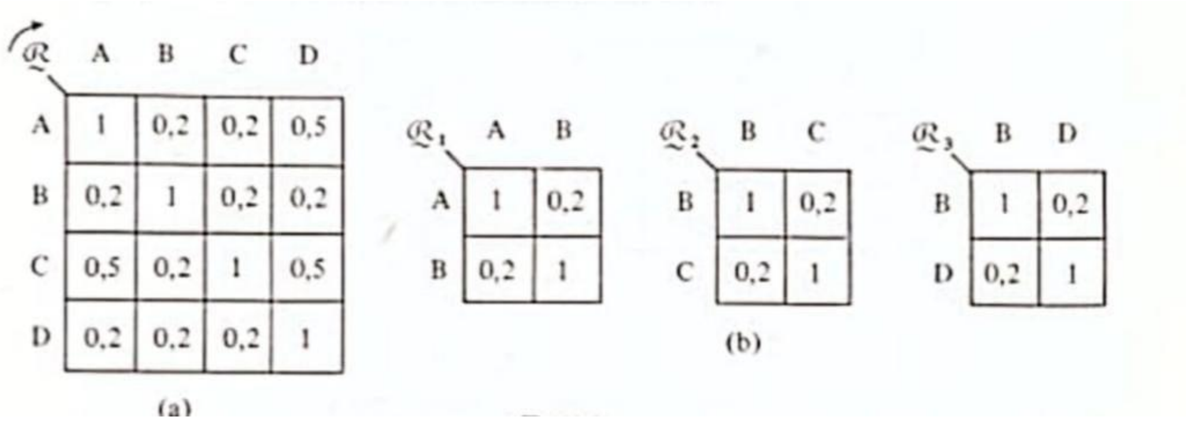


Fig. 20.2

nonsymmetric fuzzy relation; we shall see in Section 23 that this is a relation of fuzzy order.

Example 2. Figure 20.3a represents a fuzzy preorder relation. We may find three similitude subrelations $\underset{\sim}{\mathcal{R}}_1, \underset{\sim}{\mathcal{R}}_2$ and $\underset{\sim}{\mathcal{R}}_3$, (Figure 20,3b); but if these are maximal, they are not disjoint and thus do not constitute similitude classes,



$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	1	0,2	0,2	0,5
B	0,2	1	0,2	0,2
C	0,5	0,2	1	0,5
D	0,2	0,2	0,2	1

(a)

$\underset{\sim}{\mathcal{R}}_1$	A	B
A	1	0,2
B	0,2	1

$\underset{\sim}{\mathcal{R}}_2$	B	C
B	1	0,2
C	0,2	1

$\underset{\sim}{\mathcal{R}}_3$	B	D
B	1	0,2
D	0,2	1

(b)

Fig 20.3

Reducible fuzzy preorder. A furzy preorder decomposable into similitude classes will be called a reducible fuzzy preorder. Thus, the fuzzy preorder in Figure 20.1 is reducible, but that of Figure 20.3a is not.

The examples given above have involved finite sets E; but decomposition into similitude classes, such as that which has been explained, remains valid if E is infinite, denumerably or not. The classes may then be finite or not, and their number finite or infinite, But, of course, representations using matrices or Berge graphs may be made only. in cases where E is denumerable.

The search for maximal similitude subrelations of a preorder (E finite). In certain simple cases, by examining the pairs of elements for which one has symmetry, one obtains. immediately the maximal similitude subrelations, which may or may not be disjoint. But it is convenient to have at one's disposal a general procedure. We give in Appendix B, page 387, some appropriate algorithms.

21. ANTISYMMETRY

A fuzzy binary relation is *antisymmetric* if

$\forall(x,y) \in E \times E$ with $x \neq y$:

(21.1) $$\left(\mu_{\underline{\mathcal{R}}}(x,y) \neq \mu_{\underline{\mathcal{R}}}(y,x)\right) \; or \; \left(\mu_{\underline{\mathcal{R}}}(x,y) = \mu_{\underline{\mathcal{R}}}(y,x) = 0\right)$$

Examples. Figures 22.1-22.3 give some examples of antisymmetric fuzzy binary relations. Thus (Figure 22.1),

(21.2) $$\mu_{\underline{\mathcal{R}}}(A,B) < \mu_{\underline{\mathcal{R}}}(B,A),$$
$$\mu_{\underline{\mathcal{R}}}(A,C) = \mu_{\underline{\mathcal{R}}}(C,A) = 0.$$

$\underline{\mathcal{R}}$	A	B	C	D	E
A	0	0,3	0	0,9	1
B	0,5	0,8	0,6	0,8	0
C	0	0,5	1	0	1
D	0,5	1	0,2	1	0,3
E	0	0	0	0,2	0

Fig. 21.1

$\underline{\mathcal{R}}$	A	B	C	D
A	0	0	0	0,8
B	0	0	0,6	0
C	1	0,2	0,3	1
D	1	0	0	1

Fig. 21.2

$\underline{\mathcal{R}}$	A	B	C	D	E	F
A	0	0,3	0	0,2	0	0,8
B	0	1	1	0,8	0	0,5
C	0	0	0,3	0	0	0,6
D	0	0	0	0	0	0
E	0	0,3	0	0,2	0,5	0,5
F	0,7	0	0	0	0	0,1

Fig. 21.3

(21.2) $$\mu_{\underline{\mathcal{R}}}(A,D) > \mu_{\underline{\mathcal{R}}}(D,A),$$
$$\mu_{\underline{\mathcal{R}}}(A,E) > \mu_{\underline{\mathcal{R}}}(E,A).$$

And so on.

Another example. Let $x\underline{\mathcal{R}}y$where $x,y \in R^+$; the relation $\underline{\mathcal{R}}$ such that

(21.3) $$\mu_{\underline{\mathcal{R}}}(x,y) = e^{-(ax+by)} \quad a > b > 1,$$

is antisymmetric.

Remark. One should not confuse a nonsymmetric graph with an antisymmetric graph. For the first, one may write

$\exists(x,y) \in E \times E$ with $x \neq y$:

(21.4) $$\mu_{\underline{\mathcal{R}}}(x,y) \neq \mu_{\underline{\mathcal{R}}}(y,x).$$

Thus, the graph in Figure 21.4 is nonsymmetric [there exists at least one ordered pair (x,y) for which (21.4) is satisfied]. But this graph is not antisymmetric [there is at least one ordered pair (x,y) for $\mu_{\underline{\mathcal{R}}}(x,y) \neq \mu_{\underline{\mathcal{R}}}(y,x) \neq 0$, for example the ordered pair (C,D)].

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E	F
A	0	0,5	0,3	1	0	0
B	0,3	0	0,4	1	0,2	0
C	0	0,4	0,6	1	0,3	0,4
D	0	0	0	1	0,2	0,6
E	0	0,2	0,3	0,3	0,7	1
F	0	0,6	0,4	0,6	1	0

Fig. 21.4

Ordinary antisymmetric graph associated with an antisymmetric fuzzy relation. To any antisymmetric fuzzy relation $\underline{\mathcal{R}}$ one will associate one (and only one) ordinary antisymmetric graph G such that

$\forall(x,y) \in E \times E$:

(21.5)
1) $x \neq y$ and $\mu_{\underline{\mathcal{R}}}(x,y) > \mu_{\underline{\mathcal{R}}}(y,x) \Rightarrow (x,y) \in G$ and $(x,y) \notin G$,
2) $x \neq y$ and $\mu_{\underline{\mathcal{R}}}(x,y) = \mu_{\underline{\mathcal{R}}}(y,x) = 0 \Rightarrow (x,y) \in G$ and $(x,y) \notin G$,

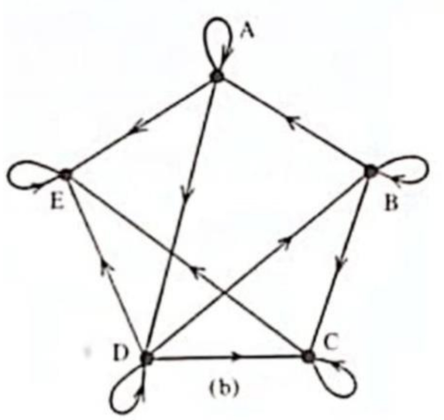
We shall take (arbitrarily) for G

(21.6) $$\forall(x,x) \in E \times E: (x,x) \in G$$

This will prove convenient later for the study of nonstrict relations of order.

Example 1. Figure 21.5 and 21.6 represent ordinary antisymmetric graphs associated with the relations in Figures 21.1 and 21.2.

G	A	B	C	D	E
A	1	0	0	1	1
B	1	1	1	0	0
C	0	0	1	0	1
D	0	1	1	1	1
E	0	0	0	0	1



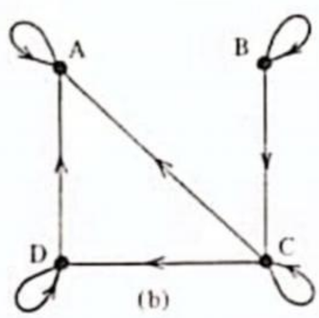
(b)

(a)

Fig.21.5

G	A	B	C	D
A	1	0	0	0
B	0	1	1	0
C	1	0	1	1
D	1	0	0	1

(a)



(b)

Fig.21.6

Example 2. We recall that the notion of an ordinary graph encompasses all ordinary sets, finite or not. Thus, to any antisymmetric fuzzy relation defined on a finite or an infinite set, one may associate an ordinary antisymmetric graph. Thus, to the fuzzy antisymmetric relation de fined by (21.3), we shall associate the ordinary graph

(21.7) $$G = \{(x, y) | y \geq x\}$$

this graph is represented in Figure 21.7.

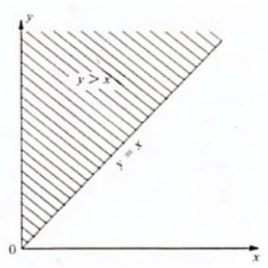


Fig. 21.7

Remark. One ought not to confuse the concept of the ordinary antisymmetric graph associated with an antisymmetric fuzzy relation with that of the ordinary graph nearest to this fuzzy relation, these two graphs have no direct relationship.

Perfect antitymmetry. L. A. Zadeh defines antisymmetry more restrictively, but in a way having some further interesting properties; we shall call this perfect antisymmetry. A perfect antisymmetric relation is one such that†

$\forall\ (x, y) \in E \times E$ with $x \neq y$:

(21.8)
$$\mu_{\underline{\mathcal{R}}}(x, y) > 0 \quad \Rightarrow \quad \mu_{\underline{\mathcal{R}}}(y, x) = 0.$$

†L. A. Zadeh gives another definition:

$$\big(\mu_{\underline{\mathcal{R}}}(x, y) > 0\ \ and\ \ \mu_{\underline{\mathcal{R}}}(y, x) > 0\big) \Rightarrow (x = y).$$

We shall return later to several interesting properties of perfect antisymmetry in a discussion of the idea of perfect ordet,

Remark. Any perfect antisymmetric relation is evidently antisymmetric.

Example 1. Figure 21.8 represents a perfect antisymmetric relation, Figure 22.9 shows the ordinary antisymmetric graph asociated with this relation.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E	F
A	0	0,8	0,4	0,6	0	0
B	0	0,3	0	0,6	0	0,7
C	0	0,3	1	0,2	1	0,6
D	0	0	0	0,8	0	0,3
E	0	0,5	0	0	0	1
F	0,7	0	0	0	0	1

Fig. 21.8

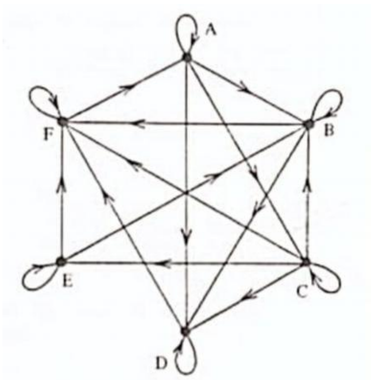


Fig. 21.9

Example 2. Consider the two domains$D_1^{\dagger}$andD_2 of $R^+ + R^+$ indicated in Figure 21.10. The relation $x\underline{\mathcal{R}}y$ defined on R^+,

(21.9)
$$\mu_{\underline{\mathcal{R}}}(x, y) \begin{array}{ll} = \mu_1(x, y) & if\ (x, y) \in D_1 \\ = \mu_2(x, y) & if\ (x, y) \in D_2 \\ = 0 & if\ (x, y) \notin D_1 \cup D_2 \end{array}$$

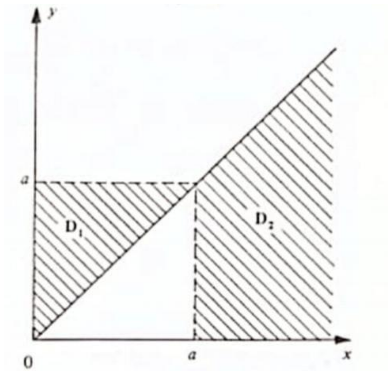


Fig. 21.10

Is a perfect antisymmetric relation. Moreover, Figure 21.10 represenrs the ordinary antisymmetric graph associated with the expression (21.9).

22. FUZZY ORDER RELATIONS

A binary relation that is

(1) reflexive [according to (16.7)]
(2) transitive [according to (16,8) or (16.9)]
(3) antisymmetric [according to (22,1)]

is a *fuzzy order relation* (we shall also say simply *order relation* if no confusion in possible).

One may also define this property in the following fashion: A fuzzy preorder relation that is antisymmetric† is a furzy order relation.

Example 1. Figures 23.1 and 23.2 represent fuzzy order relations. One may verify that these are indeed reflexive. transitive, and antisymmetric.

$\underset{\sim}{R}$	A	B	C	D
A	1	0,8	0	0
B	0,2	1	0	0
C	0,3	0,4	1	0,1
D	0	0	0	1

Fig 22.1

Exemple 2. The relation defined by (19.14) and Figure 19.2 is a fuzzy order relation.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	1	0,8	0,8	0,8
B	0,5	1	0,6	1
C	0,5	1	1	1
D	0,5	0,6	0,6	1

Fig 22.2

Example 3. The relations $x\underline{\mathcal{R}}y$ where $x, y \in N$ (Figure 22.3) is a fuzzy order relation.

†This is then reducible and each similitude class contains only one element.

$\underset{\sim}{\mathcal{R}}$	0	1	2	3	4	5	
0	1	e^{-1}	e^{-2}	e^{-3}	e^{-4}	e^{-5}	...
1	0	1	e^{-3}	e^{-4}	e^{-5}	e^{-6}	...
2	0	0	1	e^{-5}	e^{-6}	e^{-7}	...
3	0	0	0	1	e^{-7}	e^{-8}	...
4	0	0	0	0	1	e^{-9}	...
5	0	0	0	0	0	1	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 22.3

Theorem I Every fuzzy order relation induces an order (in the sense of the theory of sets) on its reference set through the relation

(22.1) $$\mu_{\underline{\mathcal{R}}}(x, y) \geq \mu_{\underline{\mathcal{R}}}(y, x)$$

This order will be denoted$y \succcurlyeq x$

Proof. It suffices to consider the ordinary antisymmetric graph associated with the fuzzy order relation.

Examples. Figures 22.4 and 22.5 represent, respectively, the ordinary antisymmetric graphs associated with the fuzzy order relations given in Figures 22.1 and 22.2.

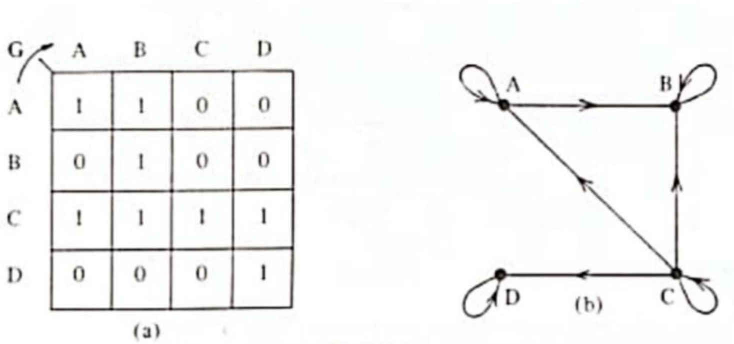


Fig 22.4

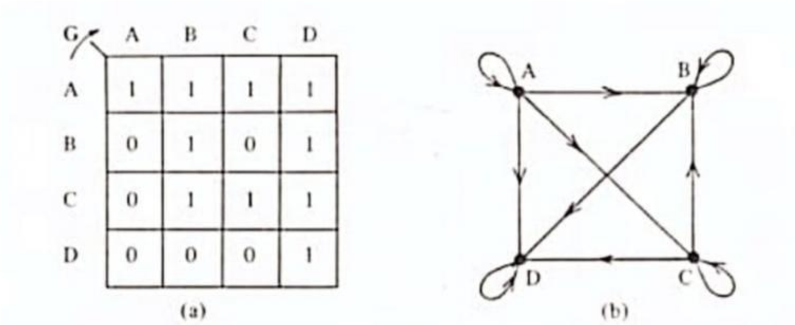


Fig. 22.5

Figure 22.6 represents the denumerably infinite ordinary graph associated with the relation presented in Figure 22.3.

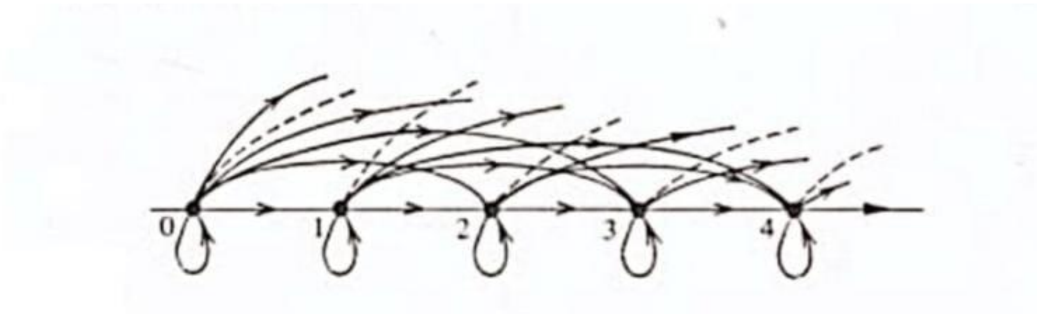


Fig 22.6

Fuzzy relation of total order.† A fuzzy relation is of total order (a totally ordered fuzzy relation) if its associated ordinary graph represents a total order.

An example is given in Figure 22.5. Using the notation

(22.2) $$y \geqslant x \quad if \quad \mu_{\underline{\mathcal{R}}}(x, y) > \mu_{\underline{\mathcal{R}}}(y, x)$$

that is, if$(x, y) \in \ G$ and $(y, x) \in G$, one then has

(22.3) $$D \geqslant B \geqslant C \geqslant A$$

Partially ordered fuzzy relations. A fuzzy relation is of partial order (a partially ordered fuzzy relation) if its associated ordinary graph is partially ordered, that is, is ordered but not totally ordered.

This is the case in the example of Figure 22,4, One has,

(22.4) $$B \succcurlyeq A \succcurlyeq C, \quad D \succcurlyeq C.$$

†This is called a linear order relation by L. A. Zadeh when this order is perfect. One may define a linear order with the more restrictive condition of antisymmetry:

$$x \neq y, \mu_{\underset{\sim}{\mathcal{R}}}(x,y) > 0 \quad or \quad \mu_{\underset{\sim}{\mathcal{R}}}(y,x) > 0$$

(exclusive)

Perfect order relations. If one takes the notion of perfect antisymmetry[according to (21.8)] in place of the notion defined by (21.1), one will then have a perfect order relation.

All of these order relations have particularly interesting properties, which we shall examine later,

Nonstrict and strict order relations. As in the theory of ordinary sets, one may distinguish between nonstrict (transitive, reflexive, antisymmetric) order relations and strict (transitive, reflexive, antisymmetric) order relations. A nonstrict order relation will generally be called an *order relation*, and a strict order relation will have to be made precise by its adjective. Such a relation may also be called a *nonreflexive order relation*.

A nonstrict order being denoted, as we have indicated,

(22.5) $$y \succcurlyeq x$$

then a strict order will be denoted

(22.6) $$y \succ x$$

We shall give several examples of fuzzy order relations that are strict.

Example 1. Figure 22.7 gives an example of a strict order relation; it is also a perfect order relation. Further, the order is total. One may verify that one has

(22.7) $$A < B < C < D$$

$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	0	0,8	0,7	0,7
B	0	0	0,6	0,4
C	0	0	0	0,5
D	0	0	0	0

Fig. 22.7

Example 2. Consider $x\underline{\mathcal{R}}y$, where $x, y \in R$ and

$$\mu_{\underline{\mathcal{R}}}(x,y) \begin{array}{l} = 0\underline{\mathcal{R}}y < x, \\ = \frac{1}{1+\frac{1}{(x-y)^2}}, y \geq x \end{array} \tag{22.8}$$

This is a relation of strict and perfect order [one may check that for $y = x, \mu_{\underline{\mathcal{R}}}(x,y) = 0$. Further, the order is total. This fuzzy relation may be taken to represent (rather poorly) the proposition $y > x$.

Important general remark. All definitions associated with order relations in ordinary sets† are directly transposable to fuzzy order relations, it is sufficient to pass through the notion of the associated ordinary graph. It is thus that one may study for fuzzy onder relations the classical concepts:

greatest and least element;

majorant and minorant;

limit superior and limit inferior;

maximal chain;

filtering set;

Hasse diagram;

semilattice and lattice.

We shall take up again, when necessary, certain of these concepts for particular uses.

We return now to the concept of a reducible fuzzy preorder for an important theorem

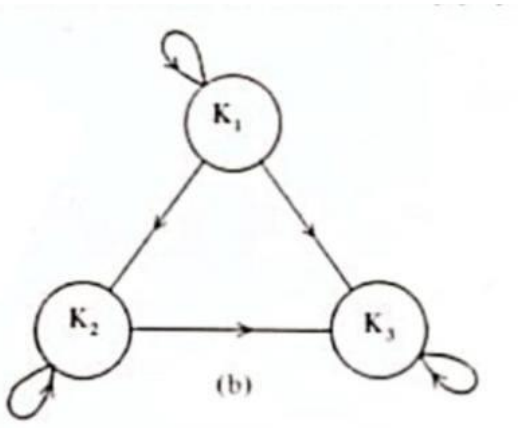
Theorem II. In a reducible fuzzy preorder relation, there exists at least one similitude class, and the similitude classes form among themselves a fuzzy order relation if one considers the concept of the strongest path from one class to another.

Proof. The relation formed by the similitude classes in necessarily antisymmetric, otherwise certain classes would not be disjoint

Example 1. Return now to the example of Figure 21.2. For the order relation between these classes, one has the graph in Figure 23.5 for the associated ordinary graph.

G	K_1	K_2	K_3
K_1	1	1	1
K_2	0	1	1
K_3	0	0	1

(a)



(b)

Fig. 22.8

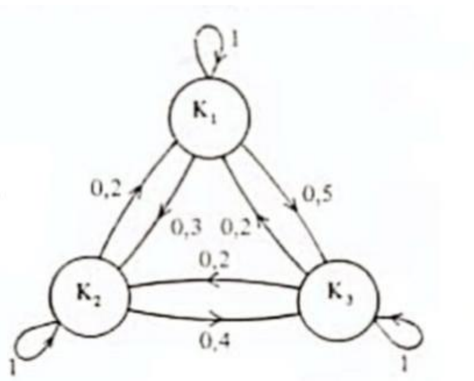
Thus, for these classes, there exists a total order

(22.9) $$K_3 \succ K_2 \succ K_1$$

Having thus presented the ordinary graph constituting the ordinary order relation between the similitude classes, one may then give the fuzzy order relation existing between the classes, obtaining the relation determined by the strongest path existing between each class For the example of Figures 20,1, 20.2, and 22.8, these results have been given in Figure 20.2; similarly, we reproduce here (Figure 22.9) those holding for Figure 22.8.

†For all that omnceres definitions with respect to the theory of (ordinary) sets, we refer to the work of Kaufmann and Precigout [K]

G	K_1	K_2	K_3
K_1	1	0,3	0,5
K_2	0,2	1	0,4
K_3	0,2	0,2	1



(a) (b) Fig. 22.9

In the case of the example presented in Figure 21.1 determination of the strongest paths existing between K_1andK_2, $K_2$and K_3.and finally between K_2 and K_3 is very easy. The class $K_1 = \{A, B, C, D, E, F\}$ and the class $K_2 = \{D\}$ are joined by paths whose values are given by

A	0,3
B	0,3
C	0,3
E	0,3
F	0,3

The strongest path (not unique) has value 0.3. For K_2toward K_1. one sees

	A	B	C	E	F
D	0,2	0,2	0,2	0,2	0,2

and the nonunique strongest path has value 0.2.

For K_1toward $K_3 = \{G\}$, one may see

A	0,4
B	0,5
C	0,5
E	0,5
F	0,5

and the nonunique strongest path has value 0,5. In the same manner one finds that the nonunique strongest path for K_3 toward K_1 has value 0,2.

One also obtains 0,4 for $K_2 \rightarrow K_3$ and 0,2 for $K_3 \rightarrow K_2$, where the determination istrivial since these classes each have only one unique element. It is thus that Figure 22.9has been obtained.

More generally, to construct the fuzzy order relation existing between the classes, one proceeds in the following fashion:

(1) Find the similitude classes K_i in the reducible fuzzy preorder. For these, consider the ordered pairs (x, y) for which one has

$$\mu_{\underline{\mathcal{R}}}(x, y) = \mu_{\underline{\mathcal{R}}}(y, x)$$

With respect to these ordered pairs construct the maximal similitude subrelations†. If these are all disjoint, one has obtained the similitude classes. If there exist at least two that are not disjoint, we do not have a reducible fuzzy preorder.

(2) For each ordered pair $(K_i, K_j), i \neq j$, examine the fuzzy subrelation$\underline{\mathcal{R}}$if existing between $K_i$and K_j (rows of K_i, and columns of K_j . Determine the global projection of $\underline{\mathcal{R}}_{ij}$ [see (12.13)]; thus,

$$h(\underline{\mathcal{R}}_{ij}) = \bigvee_{x} \bigvee_{y} \mu_{\underline{\mathcal{R}}_{ij}}(x, y), x \in K_1, y \in K_2 \,.$$

(4) Assign the value $h(\underline{\mathcal{R}}_{ij})$ to the membership function of the pair (K_i, K_j).

		K_1		K_2			K_3	K_4		K_5
	$\underline{\mathcal{R}}$	B	D	H	A	C	E	F	G	I
K_1	B	1	0,7	0,7	0,7	0,7	0,4	0,7	0,7	0,1
	D	0,7	1	0,8	0,8	0,9	0,4	0,7	0,8	0,1
K_2	H	0	0	1	0,8	0,6	0,3	0	0	0
	A	0	0	0,8	1	0,6	0,3	0	0	0
	C	0	0	0,6	0,6	1	0,3	0	0	0
K_3	E	0	0	0,2	0,2	0,2	1	0	0	0
K_4	F	0	0	0	0	0	0	1	0,3	0
	G	0	0	0	0	0	0	0,3	1	0
K_5	I	0	0	0	0	0	0	0	0	1

Fig. 22.10

†Use if necessary one of the algorithms given in Appendix B. p. 387.

Example 2. The example given in Figure 22.10 is a little more complicated. One may notice in this reducible fuzzy preorder several particularities that have not appeared in the preceding examples. The existence of a partial order between the classes is evident in Figure 22.11. In the preceding example, we had a total order (see Figure 22,8).

$\underset{\sim}{\mathcal{R}}_K$	K_1	K_2	K_3	K_4	K_5
K_1	1	0,9	0,4	0,8	0,1
K_2	0	1	0,3	0	0
K_3	0	0,2	1	0	0
K_4	0	0	0	1	0
K_5	0	0	0	0	1

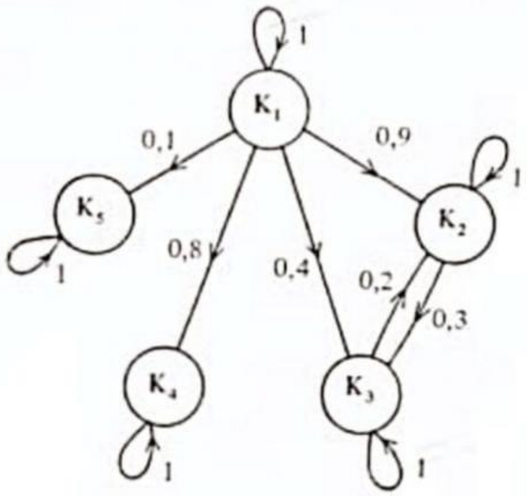


Fig. 22.11

Example 3. The fuzzy relation presented in Figure 23.12 is a reducible fuzzy pre order relation if one imposes

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots \leq 1$$

We may see that this decomposes into an infinity of similitude classes forming among themselves a total order

$$C_1 \prec C_2 \prec C_3 \prec \cdots$$

	C_1		C_2		C_2		C_3		
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	...
x_1	1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	...
x_2	a_1	1	a_2	a_2	a_2	a_2	a_2	a_2	...
x_3	0	0	1	a_3	a_3	a_3	a_3	a_3	...
x_4	0	0	a_3	1	a_4	a_4	a_4	a_4	...
x_5	0	0	0	0	1	a_5	a_5	a_5	...
x_6	0	0	0	0	a_5	1	a_6	a_6	...
x_7	0	0	0	0	0	0	1	a_7	...
x_8	0	0	0	0	0	0	a_7	1	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 22.12

23. ANTISYMMETRIC RELATIONS WITHOUT CIRCUITS, ORDINAL RELATIONS, ORDINAL FUNCTION IN A FUZZY ORDER RELATION

We shall consider a fuzzy relation (E finite) that possesses the three properties:

(1) reflexivity [according to (16.7)

(2) antisymmetry (according to (22.1)]

(3) does not have an ordinary circuit in its associated ordinary graph, other than loops, that is, other than circuits of length 1, such as (x, x).

Such a relation will be called a *fuzzy ordinal relation*,

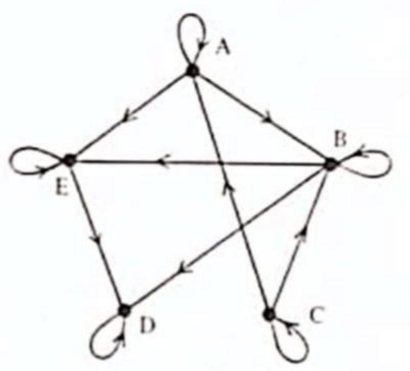
Example 1. The furzy relation in Figure 24,1 is an ordinal relation. By con structing the associated ordinary graph (Figure 24,2), one may verify that this relation is indeed reflexive, antisymmetric, and without circuits other than loops.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	1	0,8	0	0	0,7
B	0,4	1	0,6	1	0,3
C	0,9	0,8	1	0	0
D	0	0	0	1	0
E	0,2	0	0	0	1

Fig. 23.1

G	A	B	C	D	E
A	1	1	0	0	1
B	0	1	0	1	1
C	1	1	1	0	0
D	0	0	0	1	0
E	0	0	0	0	1

(a)



(b)

Fig. 23.2

Review of the notion of the ordinal function of an ordinary antixymmetric finite graph without circuits. We shall consider an ordinary graph without circuits $G \subset E \times E$, E finite. We shall again designate G by the ordered pair(E, Γ), where $E \underset{\Gamma}{\rightsquigarrow} E$, Γ representing the mapping of Einto E, which in general is multivalued.

We define the ordinary subsets$N_0, N_1, \ldots, N_r$, such that†

†Some authors prefer to define the ordinal function of an ordinary graph by replacing the inverse mapping I with the direct mapping I in formulas (23.1). This has the effect of eventually giving another order of levels

(23.1)
$$N_o = \{X_i | \Gamma^{-1}\{X_i\}\} = \phi\},$$
$$N_1 = \{X_i | \Gamma^{-1}\{X_i\}\} \subset N_0\},$$

$$N_2 = \{X_i | \Gamma^{-1}\{X_i\}\} \subset N_0 \cup N_1\},$$
$$\vdots$$
$$N_r = \{X_i | \Gamma^{-1}\{X_i\}\} \subset \cup_{k=0}^{r-1} N_k\}.$$

Where$r$ is the least integer such that

(23.2) $$\Gamma N_r = \phi$$

One may easily show that the ordinary subsets $N_k, k = 0,1,2, \dots, r$form a partition of E and are totally and strictly ordered by the relation

(24.3) $$N_k \prec N_k' \quad \Leftrightarrow \quad k < k'$$

The function $O(X_j)$ defined by

(23.4) $$X_i \in N_k \quad \Leftrightarrow \quad O(X_i) = k$$

is called the *ordinal function of an ordinary graph without circuits.*

In other words, less precise but more concise: One has in mind the decomposition of the set of vertices of the ordinary graph G without circuits into ordinary subsets, dis joint and ordered so that if one of these vertices belongs to one of the subsets carrying the number k, all vertices following the vertex being considered must be placed in a subset carrying a number larger than k.

The ordinary subsets of the partition are called *levels.*†

Example. The ordinary graph without circuits in Figure 23.3 has been decomposed into levels in Figure 24.4. If X_i, is a vertex of the graph, to each X_i, there corresponds an N_k or more simply a $k \in \{0,1,2, \dots, 5\}$. The function $X_i \rightsquigarrow k$ represented in Figure 24.5 is the ordinal function of the graph. An enumeration of the vertices is presented in Figure 24.6.

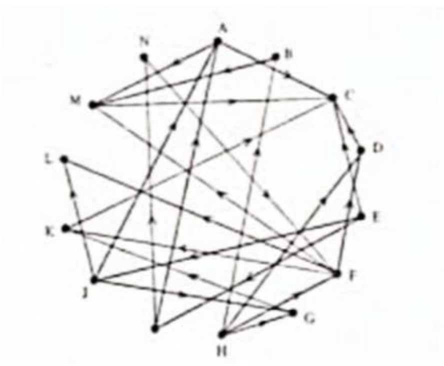
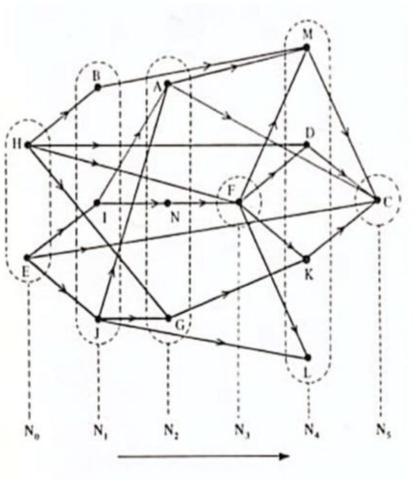


Fig. 23.3

† some authors call these ordinary subsets ranks

The ordinal function of a graph is not in general unique, it may be defined with respect to the largest elements of the ordered set instead of with respect to the smallest, that is to say, ordered from right to left instead of toward the right as we have done in the examples of Figure 23.3-23.6.



function f

Vertex	Level
A	2
B	1
C	5
D	4
E	0
F	3
G	2
H	0
I	1
J	1
K	4
L	4
M	4
N	2

Fig. 23.4 Fig. 23.5

The notion of ordinal function plays an important role in a large number of theoretical combinatorial problems and practical applications,

Extension of the notion of ordinal function to an ordinary graph having circuits. For this, it suffices to consider equivalence classes (with respect to the relation, "there exists a path from X_i to X_j and vice versa") of the ordinary graph.

These classes are the maximal ordinary subsets for the equivalence relation. These classes then form an order (total or partial, depending on the case). If the order is total, one has the ordinal function; if it is partial, one will seek the ordinal function of the ordinary graph without circuits formed by these classes,

†In the sense given to the words greatest element and least element in the theory of ordinary ordered sets

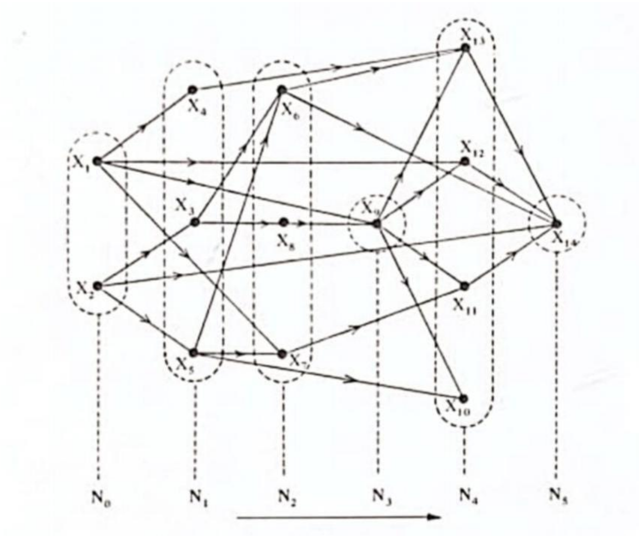


Figure 23.6

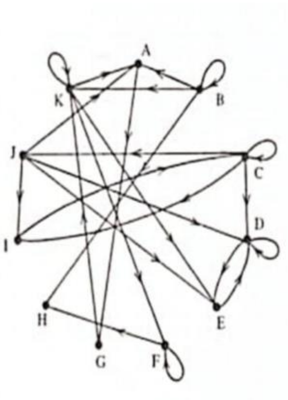


Fig. 23.7

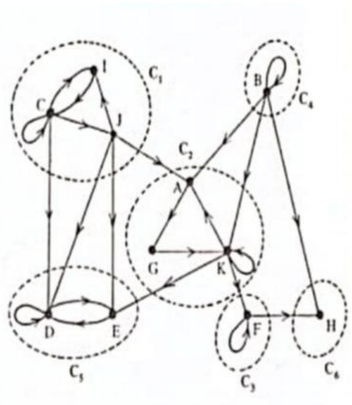


Fig. 23.8

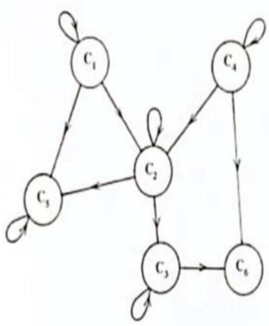


Fig. 23.9

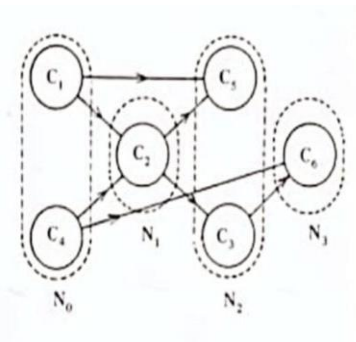


Fig. 23.10

An example is given in Figures 23.7-23.10.

Method† for determining the level of a graph without circuits. We shall consider the Boolean matrix of the ordinary graph in Figure 24.3, this matrix is presented in Figure

23.11. We form a row $\Lambda_0$in which appears the sum of the rows of the matrix ‡. The zeros of $\Lambda_0$give the vertices that are not precedents of one another, thus E and H form level N_0.Eliminating the sum of the E and$H$ rows from the $\Lambda_0$row, one obtains the rowΛ_1.Where the zeros of the row Λ_0 have been replaced by a X (cross). The zeros that appear in the row Λ_1 give the vertices that are not precedents of one another whenever E and Hhave been eliminated; these are $B, I,$ and J, which form N_1 We eliminate from row Λ_1 the sum of rows B, I, and J after having replaced all the zeros previously appearing with an X; the new zeros that appear in $\Lambda_2$give the vertices that are not precedents of one another whenever E, H, B, I, and J have been eliminated, these are $A, G,$ and N, which form N_2. And we continue thus until exhaustion. Afterward, it remains only to construct the ordinary graph (Figure 24.4) where the vertices appear with their respective levels. An arbitrary enumeration of the vertices is represented in Figure 24.6; it respects the ordinal function.

When the graph contains at least one circuit, there exists a row Λ_i in which it is impossible to make a new zero appear. This also therefore gives us an automatic means of checking whether a graph is without circuits.

If one has seen how to obtain the ordinal function in the inverse sense, by taking the greatest elements of the order (that is, from the right to the left in our representation). one may utilize exactly the same procedure by taking the transpose of the Boolean matrix

†A method due to M. Democron of Honeywell Butt Cie., Paris,

‡That is, the sum calculated in each column

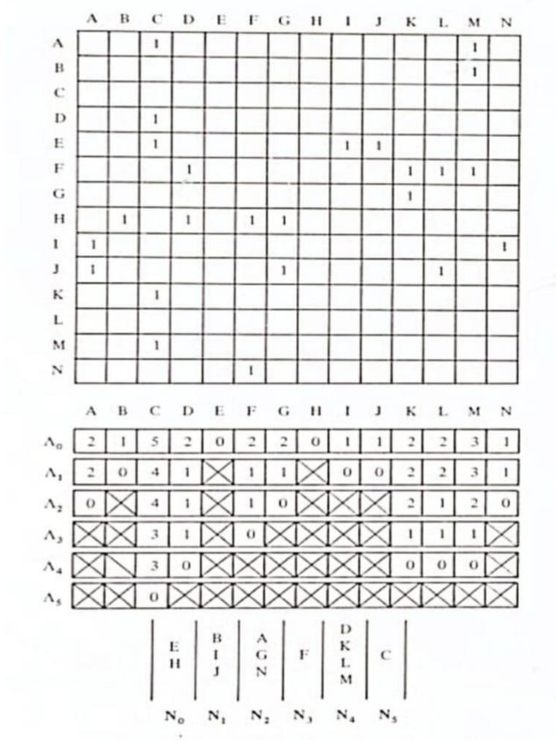


Fig. 23.11

(the rows become the columns and vice versa). Thus, reconsidering the example in Figures 24.3-24.6, we seek this time an ordinal function from right to left. The result is presented in Figure 24.12.

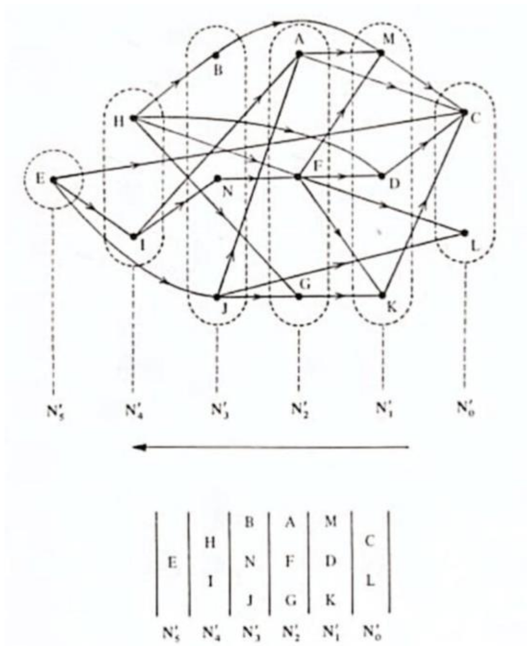


Fig. 23.12

Ordinal function of a fuzzy order relation. An order relation is an ordinal relation; it is reflexive, antisymmetric, and without circuits; it is moreover transitive. One may then define an ordinal function for it.

An example will serve to illustrate,

In Figure 23.13 we have presented a fuzzy order relation that constitutes a partial order. In Figure 23.14 we have presented the ordinal function of the associated ordinary graph with respect to the smallest elements. In this graph we have intentionally omitted the loops

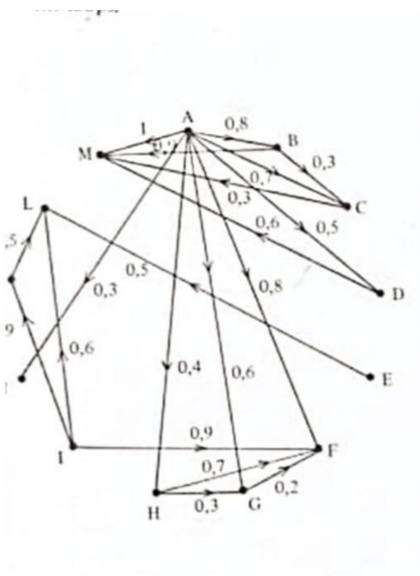


Fig. 23.13

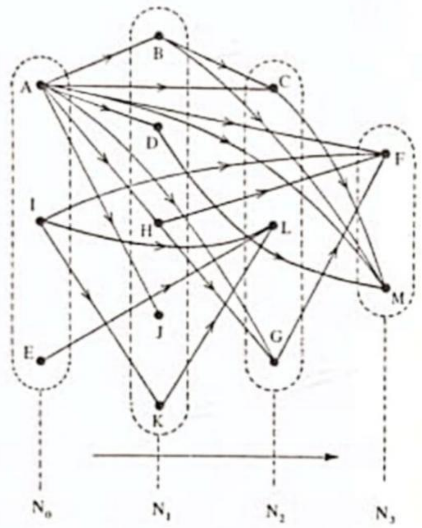


Fig. 23.14

We now consider the fuzzy order relation† presented in Figure 23.15a. Its associated ordinary graph has been given in Figure 23.15b. By permuting the elements in a manner to satisfy the ordinal function given in Figure 23.14 (Figures 23.14 and 23,156 represent the same ordinary graph), one sees a triangular matrix appear. By reconsidering the fuzzy order relation in the total order of its elements chosen to conform to the ordinal function, one obtains a fuzzy order relation that will be said to be triangular (Figure 23.15d). One knows that it is important, for whatever calculations, to know how to reduce a matrix to triangular form.

†We have taken as an example a perfect order relation with the desire of presenting a simple example: but the considerations that follow would remain valid for a fuzzy order relation that is not perfect, and the property that gives a triangular matrix in verified only for ordered pairs (x, y) such that$\mu_{\underline{\mathcal{R}}}(x, y) > \mu_{\underline{\mathcal{R}}}(y, x)$

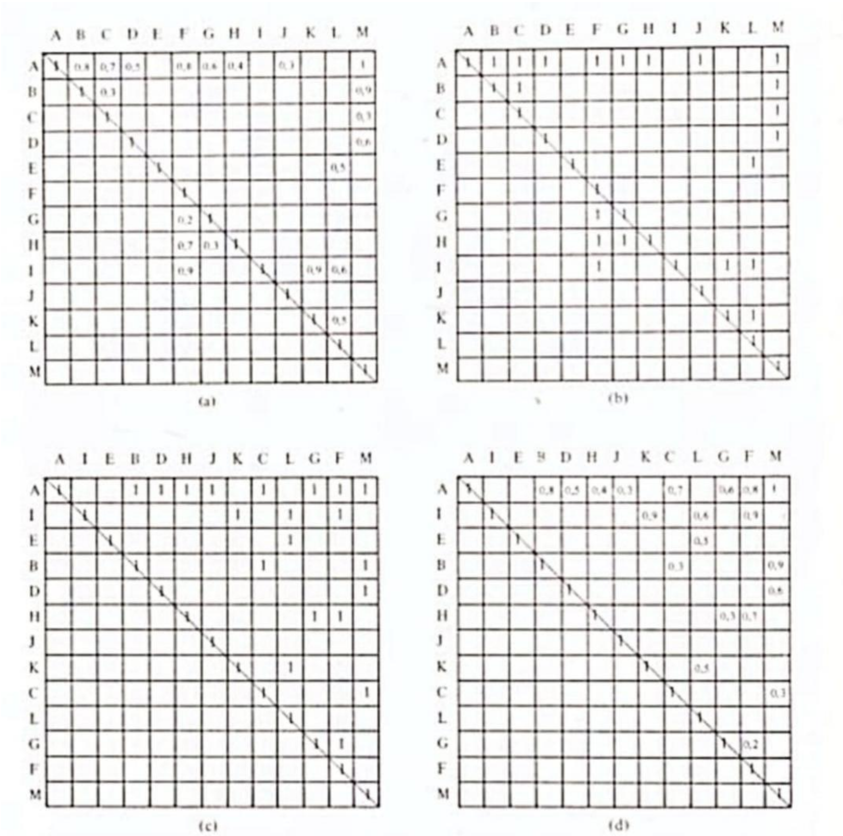


Fig. 23.5. In these matrices, the empty squares correspond to zeros.

Utility of the notion of ordinal function in fuzzy preorder relations. We have seen in Section 22 that the notion of similitude class induces in a fuzzy preorder relation an order (total or partial) of the similitude classes (in the case of a reducible preorder).

The associated ordinary graph of this order is evidently reflexive and antisymmetric, it is also transitive. If the preorder is an order, it may be reduced, as we have just seen, to a triangular form in its matrix representation. If the preorder is not an order, it may thenalways be reduced to a block-triangular form. Such a block-triangular form has already been presented in the example given in Figure 23.10, which we reproduce here associated with its Boolean matrix (Figure 24.16) in order to show that it is a block-triangular form.

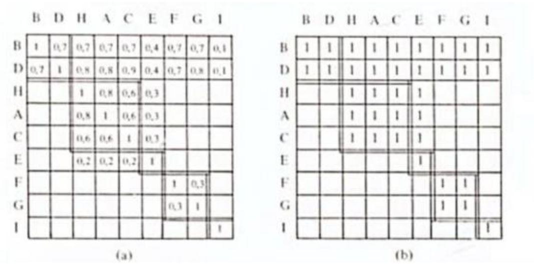


Fig. 23.16. The empty squares represent zeros.

Further, the construction of the ordinal function permits the automatic realization of the search for the Hasse diagram † corresponding to the order relation and to the deter mination of the levels of this diagram.

24. DISSIMILITUDE RELATIONS

We shall consider a similitude relation such as that defined in Section 20. For convenience, we tecall here the three properties of similitude:

(24.1) 1) $\forall (x,y),(y,z) \in E \times E$:

$$\mu_{\underline{\mathcal{R}}}(x,z) \geq \bigvee_{y} [\mu_{\underline{\mathcal{R}}}(x,y) \wedge \mu_{\underline{\mathcal{R}}}(y,z)], \qquad transitivity$$

(24.2) 2) $\forall (x,x) \in E \times E$: $\mu_{\underline{\mathcal{R}}}(x,x) = 1$ $\qquad reflexivity$

(24.3) 3) $\forall (x,y) \in E \times E$: $\mu_{\underline{\mathcal{R}}}(x,y) = \mu_{\underline{\mathcal{R}}}(x,y)$ $\qquad symmetry$

Now we associate with $\underline{\mathcal{R}}$ a relation $\underset{\underline{\mathcal{R}}}{\leftarrow}$ such that

$$\forall (x,y) \in E \times E: \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,y) = 1 - \mu_{\underline{\mathcal{R}}}(x,y) \tag{24.4}$$

†Those who are not familiar with what is called a Hase diagram in the nedinary theory of sets may consult, for example, references $[K_1, K_2]$.

Knowing that$\underline{\mathcal{R}}$ has properties (24.1)-(24.3), these are then properties of$\underline{\mathcal{R}}$. Beginning with (24.1), one has

$$1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,y) \geq \bigvee_{y} \left[1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,y)\right] \wedge \left[1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(y,z)\right]. \tag{24.5}$$

But, according to (7.32),

$$\left[1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,y)\right] \wedge \left[1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(y,z)\right] = 1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,y) \vee \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(y,z). \tag{24.6}$$

Thus (24.5) may be written

$$1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,z) \geq \bigvee_{y} \left[1 - \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,y) \vee \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(y,z)\right] \tag{24.7}$$

Or

$$\mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,z) \leq \bigwedge_{y} \left[\mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(x,y) \vee \mu_{\underset{\underline{\mathcal{R}}}{\leftarrow}}(y,z)\right] \tag{24.8}$$

This property will be called $min - max\ transitivity$. †

Concerning (24.2), one may see

(24.9) $$\mu_{\underline{\overleftarrow{\mathcal{R}}}}(x,x) = 1 - \mu_{\underline{\overleftarrow{\mathcal{R}}}}(x,x) = 1 - 1 = 0.$$

And finally, symmetry is preserved. Thus we have

(24.10) 1) $\forall (x,y),(y,z),(z,x) \in E \times E$:

$$\mu_{\underline{\overleftarrow{\mathcal{R}}}}(x,z) \leq \bigwedge_{y} \left[\mu_{\underline{\overleftarrow{\mathcal{R}}}}(x,y) \vee \mu_{\underline{\overleftarrow{\mathcal{R}}}}(y,z)\right], \qquad mini-max\ transitivity$$

(24.11) 2) $\forall (x,x) \in E \times E$: $\mu_{\underline{\overleftarrow{\mathcal{R}}}}(x,x) = 0$ $\qquad antireflexivity$

(24.12) 3) $\forall (x,y) \in E \times E$: $\mu_{\underline{\overleftarrow{\mathcal{R}}}}(x,y) = \mu_{\underline{\overleftarrow{\mathcal{R}}}}(y,x)$ $\qquad symmetry$

A fuzzy binary relation that possesses properties (24.10)-(24.12) is called a dissimilitude relation,

Example 1. Figure 24.1 represents dissimilitude relation (it is, moreover, the relation $\underline{\mathcal{R}}$ corresponding to the similitude relation $\underline{\mathcal{R}}$ presented in Figure 19.1). As an exercise, we verify (24.10) for several pairs of elements,

$\mathcal{R}$	A	B	C	D	E
A	0	0,2	0,3	0	0,1
B	0,2	0	0,3	0,2	0,2
C	0,3	0,3	0	0,3	0,3
D	0	0,2	0,3	0	0,1
E	0,1	0,2	0,3	0,1	0

Fig. 24.1

†One may also call thin min-max cotransitivity.

Arc (A,B)

$$\mu(A,A) \vee \mu(A,B) = 0 \vee 0{,}2 = 0{,}2\,,$$
$$\mu(A,B) \vee \mu(B,B) = 0{,}2 \vee 0 = 0{,}2\,,$$
$$\mu(A,C) \vee \mu(C,B) = 0{,}3 \vee 0{,}3 = 0{,}3\,,$$
$$\mu(A,D) \vee \mu(D,B) = 0 \vee 0{,}2 = 0{,}2\,,$$
$$\mu(A,E) \vee \mu(E,B) = 0{,}1 \vee 0{,}2 = 0{,}2\,,$$

(24.13) $$MIN[0{,}2\ ,\quad 0{,}2\ ,\quad 0{,}3\ ,\ 0{,}2\ ,\ 0{,}2] = 0{,}2.$$

$$\mu(A,B) = 0{,}2 \leq 0{,}2.$$

Arc (A,C)

$$\mu(A,A) \vee \mu(A,C) = 0 \vee 0{,}3 = 0{,}3\ ,$$
$$\mu(A,B) \vee \mu(B,C) = 0{,}2 \vee 0{,}3 = 0{,}3\ ,$$
$$\mu(A,C) \vee \mu(C,C) = 0{,}3 \vee 0 = 0{,}3\ ,$$
$$\mu(A,D) \vee \mu(D,C) = 0 \vee 0{,}3 = 0{,}3\ ,$$
$$\mu(A,E) \vee \mu(E,C) = 0{,}1 \vee 0{,}3 = 0{,}3\ ,$$
$$MIN[0.3, \dots,] = 0{,}3\ .$$
$$\mu(A,C) = 0{,}3 \leq 0{,}3\ .$$

And so on

Example 2. The relation presented in Figure 25.2 is a dissimilitude relation if

(24.14) $$1 \geq b_1 \geq b_2 \geq \cdots \geq b_i \geq \cdots \geq 0.$$

$\underset{\sim}{\mathcal{R}}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\dots$
x_1	0	b_1	b_1	b_1	b_1	b_1	b_1	$\dots$
x_2	b_1	0	b_2	b_2	b_2	b_2	b_2	$\dots$
x_3	b_1	b_2	0	b_3	b_3	b_3	b_3	$\dots$
x_4	b_1	b_2	b_3	0	b_4	b_4	b_4	$\dots$
x_5	b_1	b_2	b_3	b_4	0	b_5	b_5	$\dots$
x_6	b_1	b_2	b_3	b_4	b_5	0	b_6	$\dots$
x_7	b_1	b_2	b_3	b_4	b_5	b_6	0	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fig.24.2

This relation has been obtained from that presented in Figure 20.3 by setting

(24.15) $$\mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y) = 1 - \mu_{\underline{\mathcal{R}}}(x,y)\ letting\ b_i = 1 - a_i, i = 1,2,3, \dots$$

Example 3. The fuzzy relation

(24.16) $$\mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y) \begin{aligned} &= 1 - e^{-k(y+1)}, && y < x, k > 1 \\ &= 0, && y = x \\ &= 1 - e^{-k(x+1)}, && y > x, k > 1 \end{aligned}$$

is a dissimilitude relation. It has been obtained from (20.3) by setting

$$\mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y) = 1 - \mu_{\underline{\mathcal{R}}}(x,y).$$

We shall see several examples, but first we recall here, in order to have them nearby, axioms (5.49)-(5.52) concerning the notion of distance between two elements of a set.

If $d(X,Y)$ is this distance between $X$and Y:

$\forall X,Y,Z \in E$, one must have

(24.17) 1) $d(X,Y) \geq 0$,

(24.18) 2) $d(X,Y) = d(Y,X)$,

(24.19) 1) $d(X,Y) * d(Y,Z) \geq d(X,Z)$.

Where$*$ is the operation considered among the distances $d(X,Y)$.

To these three conditions, one may logically introduce a fourth

$$d(X,X) = 0 \tag{24.20}$$

Then considering$\mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y)$, one has indeed, by definition, that (24.17) is satisfied since $0 \leq \mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y) \leq 1$. Relation (24.18) is satisfied [see (24.12)]. Relation (24.19) is satisfied [see (24,10), where the operation$*$ is the min-max operation]. Finally, (24.20) is verified see (24.11)]. Thus, one may put

$$d(x,y) = \mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y) \tag{24.21}$$

and consider $\mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y)$ as a distance† existing between x and y.

Min-max distance between two elements in a similitude relation. Let $\underline{\mathcal{R}}$ be a similitude relation. We shall call the min-max distance between $x$and y, $y \subset E\underline{\mathcal{R}} \subset E \times E$.

$$\mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y) = 1 - \mu_{\underline{\mathcal{R}}}(x,y) \tag{24.22}$$

Example 1. We take up again the example of Figure 19.1 (seen again in Figure 24.3). This is a similitude relation $\underline{\mathcal{R}}$. Figure 245.4 represents the dissimilitude relationassociated with that of Figure 24.3. One thus has

†In this case, one may also call $\mu_{\underline{\mathcal{R}}}(x,y)$ the codistance between x and y.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	1	0,8	0,7	1	0,9
B	0,8	1	0,7	0,8	0,8
C	0,7	0,7	1	0,7	0,7
D	1	0,8	0,7	1	0,9
E	0,9	0,8	0,7	0,9	1

Fig. 24.3

$\overline{\underset{\sim}{\mathcal{R}}}$	A	B	C	D	E
A	0	0,2	0,3	0	0,1
B	0,2	0	0,3	0,2	0,2
C	0,3	0,3	0	0,3	0,3
D	0	0,2	0,3	0	0,1
E	0,1	0,2	0,3	0,1	0

Fig. 24.4

(24.22)
$$d_{\underset{\sim}{\mathcal{R}}}(A,B) = 0{,}2\ ,$$
$$d_{\underset{\sim}{\mathcal{R}}}(A,C) = 0{,}3\ ,$$
$$d_{\underset{\sim}{\mathcal{R}}}(A,D) = 0.$$
$$\dots etc$$

Example 2. Consider again example (19.3); one then has

(24.24)
$$d(x,y) \begin{array}{ll} = e^{-k(y+1)} & y < x, k > 1 \\ = 1 & y = x \\ = e^{-k(x+1)} & y > x, k > 1 \end{array}$$

25. RESEMBLANCE RELATIONS †

A relation$\underset{\sim}{\mathcal{R}}$ of such that

(25.1)
$$\forall (x,x) \in E \times E : \mu_{\underset{\sim}{\mathcal{R}}}(x,x) = 1 \qquad (reflexivity)$$

†In the theory of ordinary sets, the fact that this binary relation does not inherit the property of transitivity has provoked an almost total disinterest on the part of mathematicians in this property (an exception being C. Flament, Analyse des structures preferentiellesintracitives, Proc, Sec, Interm Conf. OR., p. 150, 1960). Just as humerous cartoonists of all times, they have engaged in a very common error, that of believing that resemblance in transitive. Recall those caricatures that one has seen in which modified images appear one after the other, as King Louis Philippe has been transformed in countenance or the emperor Napoleon III transformed into a mackerel. The talent of these humorists must not obscure their logical error. Writing, in the sense of the theory of ordinary sets, A resembles 11. B resembles C, C resembles D,…,K resembles L, therefore A resembles L. Indeed A=L. constitutes a sequence of deductions without validity. Often enough, moreover, fate deductions of this nature are used by men in the spirit of making a

joke or by political men to make the best of the stupidity of certain voters. The sophists have a particular habit of making is believe in the existence of transitivity where its existence may well be doubted.

But, with the theory of fuzzy subsets, one mas measure several sorts of resemblance with the aidof the notion of distance in the transitive closure. The notion of similitude then constitutes the bridgeexisting between equivalence and resemblance.

(25.2) $$\forall(x,y) \in E \times E: \mu_{\underline{\mathcal{R}}}(x,y) = \mu_{\underline{\mathcal{R}}}(y,x) \qquad symmetry$$

Is called a resemblance relation.†

Example 1. Figure 25.1 gives an example of a resemblance relation,

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	1	0,1	0,8	0,2	0,3
B	0,1	1	0	0,3	1
C	0,8	0	1	0,7	0
D	0,2	0,3	0,7	1	0,6
E	0,3	1	0	0,6	1

Fig.25.1

Example 2. The relation (16.12),

(25.3) $$\mu_{\underline{\mathcal{R}}}(x,y) = e^{-k(x-y)^2}, x, y \in N$$

is not, as we have seen, transitive, but it is reflexive and symmetric, it is a fuzzy relation of resemblance,

Min-max distance in a resemblance relation. If $\underline{\mathcal{R}}$is a resemblance relation,‡ then$\underline{\mathcal{R}}$, its transitive closure, is a similitude relation. One may then define the notion of min- max distance in $\underline{\mathcal{R}}$ by that in$\underline{\hat{\mathcal{R}}}$. Thus,

(25.4) $$d_{\underline{\mathcal{R}}}(x,y) = 1 - \mu_{\underline{\hat{\mathcal{R}}}}(x,y).$$

Example 1. We reconsider the example of Figure 25.1. With the aid of the composition formula (16.3) we have calculated$\underline{\mathcal{R}}$, the transitive closure of$\underline{\hat{\mathcal{R}}}$.

$\hat{\underset{\sim}{\mathcal{R}}}$	A	B	C	D	E
A	1	0,6	0,8	0,7	0,6
B	0,6	1	0,6	0,6	1
C	0,8	0,6	1	0,7	0,6
D	0,7	0,6	0,7	1	0,6
E	0,6	1	0,6	0,6	1

Fig. 25.2

$\bar{\hat{\underset{\sim}{\mathcal{R}}}}$	A	B	C	D	E
A	0	0,4	0,2	0,3	0,4
B	0,4	0	0,4	0,4	0
C	0,2	0,4	0	0,3	0,4
D	0,3	0,4	0,3	0	0,4
E	0,4	0	0,4	0,4	0

Fig. 25.3

This result is presented in Figure 25.2. Next we have calculated of such that

†See the previous footnote.

‡The composition of with R comerves reflexivity and symmetry.

(25.5) $$\mu_{\overleftarrow{\underline{\mathcal{R}}}}(x,y) = 1 - \mu_{\underline{\hat{\mathcal{R}}}}(x,y).$$

the result is presented in Figure 25.3.

Finally one has

(25.6)
$$\begin{aligned} &\mu_{\overleftarrow{\underline{\hat{\mathcal{R}}}}}(A,B) = 0{,}4\,, \\ &\mu_{\overleftarrow{\underline{\hat{\mathcal{R}}}}}(A,C) = 0{,}4\,, \\ &\vdots \\ &\mu_{\overleftarrow{\underline{\hat{\mathcal{R}}}}}(B,D) = 0{,}4\,, \\ &\dots etc \end{aligned}$$

Example 2. Consider the resemblance relation defined by

(25.7) $$\mu_{\underline{\mathcal{R}}}(x,y) = \frac{1}{1+|x-y|}, n \in N, y \in N.$$

This relation is represented in Figure 26.4.

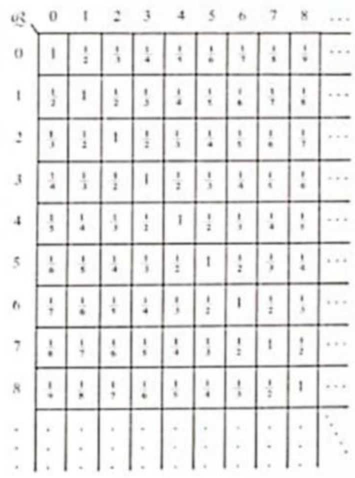
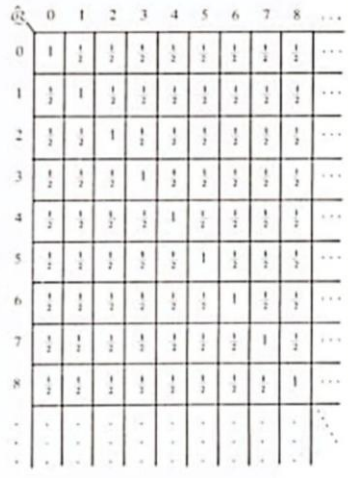
 

Fig. 25.4 Fig.25.5

Calculating†

(25.8) $$\hat{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \underline{\mathcal{R}}^3 \cup \ldots$$

one obtains the relation given in Figure 26.5. One then has

(25.9) $$\mu_{\hat{\underline{\mathcal{R}}}}(x,y) \begin{matrix} = \frac{1}{2}, x \neq y \\ = 1, x = y \end{matrix}$$

Hence, in conclusion

(25.10) $$\mu_{\underline{\mathcal{R}}}(x,y) \begin{matrix} = \frac{1}{2}, x \neq y \\ = 0, x = y \end{matrix}$$

We note that if one reconsiders (26.7) but this time with

(25.11) $$x \in R^+ \quad and \quad y \in R^+$$

one would find

(25.12) $$d_{\underline{\mathcal{R}}}(x,y) = 0$$

for all x and all y. This is not paradoxical since the distance between x and $y = x + dx$ is infinitely small and of the same order as dx. Of course, if one would give the distance some other significance than the min-max distance considered here, it would be proper to review this conclusion.

Max-product transitive closure for a resemblance relation. Let $\underline{\mathcal{R}}$be a resemblance relation. In certain cases it is preferable to measure the distance existing between elements with the aid of the max-product operation instead of the max-min operation, that is, to use (13.19) instead of (13.2); thus

(25.13) $$\mu_{\underline{\mathcal{R}}^2}(x,z) = \bigvee_y \left[\mu_{\underline{\mathcal{R}}}(x,y) \,.\, \mu_{\underline{\mathcal{R}}}(y,z)\right]$$

The max-product transitive closure of a relation is

(25.14) $$\dot{\hat{\underline{\mathcal{R}}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\dot{2}} \cup \underline{\mathcal{R}}^{\dot{3}} \cup \ldots$$

where

(25.15) $$\underline{\mathcal{R}}^{\dot{k}} = \underbrace{\underline{\mathcal{R}}\,.\,\underline{\mathcal{R}}.\ldots.\,\underline{\mathcal{R}}}_{k\ times}, \qquad k = 1,2,3,\ldots$$

The points on$\dot{\wedge}$ and $\dot{k}$ remind us that we have a max-product composition,

†In order to obtain $\hat{\underline{\mathcal{R}}}$ it is necessary to take $\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \underline{\mathcal{R}}^3 \cup \ldots$it in clear that all the elements of $\hat{\underline{\mathcal{R}}}$tend toward $\frac{1}{2}$, except these on the principal diagonal which remain equal to 1.

We see an example. Recall that for Figure 25.1, we have calculated$\hat{\underline{\mathcal{R}}}$ and $\underset{\hat{\underline{\mathcal{R}}}}{\rightarrow}$ of in Figures 25.2 and 25.3. In Figure 25.5 one may observe how we have calculated$\underline{\mathcal{R}}^{\dot{2}}, \underline{\mathcal{R}}^{\dot{3}}, \underline{\mathcal{R}}^{\dot{4}}, \underline{\mathcal{R}}^{\dot{5}}, \dot{\hat{\underline{\mathcal{R}}}}$.

$\mathcal{R}$	A	B	C	D	E
A	1	0,1	0,8	0,2	0,3
B	0,1	1	0	0,3	1
C	0,8	0	1	0,7	0
D	0,2	0,3	0,7	1	0,6
E	0,3	1	0	0,6	1

$\mathcal{R}^{2}$	A	B	C	D	E
A	1	0,3	0,8	0,56	0,3
B	0,3	1	0,21	0,6	1
C	0,8	0,21	1	0,7	0,42
D	0,56	0,6	0,7	1	0,6
E	0,3	1	0,42	0,6	1

$\mathcal{R}^{3}$	A	B	C	D	E
A	1	0,3	0,8	0,56	0,336
B	0,3	1	0,42	0,6	1
C	0,8	0,42	1	0,7	0,42
D	0,56	0,6	0,7	1	0,6
E	0,336	1	0,42	0,6	1

$\mathcal{R}^{4}$	A	B	C	D	E
A	1	0,336	0,8	0,56	0,336
B	0,336	1	0,42	0,6	1
C	0,8	0,42	1	0,7	0,42
D	0,56	0,6	0,7	1	0,6
E	0,336	1	0,42	0,6	1

$\mathcal{R}^{5}$	A	B	C	D	E
A	1	0,336	0,8	0,56	0,336
B	0,336	1	0,42	0,6	1
C	0,8	0,42	1	0,7	0,42
D	0,56	0,6	0,7	1	0,6
E	0,336	1	0,42	0,6	1

$\dot{\hat{\mathcal{R}}}$	A	B	C	D	E
A	1	0,336	0,8	0,56	0,336
B	0,336	1	0,42	0,6	1
C	0,8	0,42	1	0,7	0,42
D	0,56	0,6	0,7	1	0,6
E	0,336	1	0,42	0,6	1

Fig. 25.6

Remarks on the calculation of $\dot{\hat{\underline{\mathcal{R}}}}$. We have seen in (18.19) that

(25.16) $$\underline{\mathcal{R}} \circ \underline{\mathcal{R}} \subset \underline{\mathcal{R}} \quad = \quad \underline{\mathcal{R}}\,.\,\underline{\mathcal{R}} \subset \underline{\mathcal{R}}$$

without having the reverse be true.

Theorem II of Section 17, that is, (17.13), is also verified for the max-product. With respect to a particular k.

$$\underline{\mathcal{R}}^{\hat{k}.\dot{\hat{1}}} \circ \underline{\mathcal{R}}^{\dot{k}} \quad \Rightarrow \quad \dot{\hat{\underline{\mathcal{R}}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^{\dot{2}} \cup \underline{\mathcal{R}}^{\dot{k}} \tag{25.17}$$

And in the case where is a resemblance relation, one has likewise

$$\underline{\mathcal{R}}^{\hat{k}.\dot{\hat{1}}} = \underline{\mathcal{R}}^{\dot{k}} \quad \Rightarrow \quad \dot{\hat{\underline{\mathcal{R}}}} = \underline{\mathcal{R}}^{\dot{k}} \tag{25.18}$$

Min-sum distance in a resemblance relation. We shall call

$$\gamma_{\underline{\mathcal{R}}}(x,y) = \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,y) \tag{25.19}$$

the min-sum distance; but first we must determine whether the distance axioms (24.17-(24.20) are satisfied.

(24.17) is verified a priori since $\mu_{\hat{\mathcal{R}}}(x,y) \in [0,1]$.

(24.18) is verified a priori since the relation$\dot{\hat{\underline{\mathcal{R}}}}$ is symmetric.

(24.20) is verified a priori since the relation$\dot{\hat{\underline{\mathcal{R}}}}$is reflexive, which entails $\mu_{\hat{\underline{\mathcal{R}}}}(x,x) = 0$

It remains to show that one indeed has property (24.19). We shall operate as for(24.5)-(24.9).

One then has

$$\mu_{\dot{\hat{\underline{\mathcal{R}}}}}(x,z) \geq \bigvee_{y} \left[\dot{\hat{\underline{\mathcal{R}}}}(x,y) \,.\, \dot{\hat{\underline{\mathcal{R}}}}(x,y)\right] \tag{25.20}$$

And from here, following (8.23),

$$1 - \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,z) \geq \bigvee_{y} \left[\left[1 - \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,y)\right].\left[\,1 - \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(y,z)\right]\right] \tag{25.21}$$

$$\geq \bigvee_{y} \left[1 - \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,y) - \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(y,z) + \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,y).\mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(y,z)\right]$$

This gives

$$\mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(\,x,z) \leq \wedge_{y} \left[\mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,y) + \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(y,z) - \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,y).\mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(y,z)\right] \tag{25.22}$$

That is

$$\mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(\,x,z) \leq \wedge_{y} \left[\mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(x,y) \hat{+} \mu_{\bar{\dot{\hat{\underline{\mathcal{R}}}}}}(y,z)\right] \tag{25.23}$$

Where$\hat{+}$ is the algebraic sum defined by (12.42). Then we certainly have property (24.19) for the min-sum operation.

Example 1. Consider again the example of Figure 26.1. In Figure 26.6 we have calculated the max-product transitive closure, that is$\hat{\dot{\underline{\mathcal{R}}}}$, The min-sum distances will then be given by the relation$\vec{\hat{\dot{\underline{\mathcal{R}}}}}$ for which one has

(25.24) $$\gamma(x,y) = \mu_{\vec{\hat{\underline{\mathcal{R}}}}}(\,x,y) = 1 - \mu_{\vec{\hat{\underline{\mathcal{R}}}}}(x,y)$$

Figure 25.7 gives the min-sum distances between the various elements. Thus

$\gamma(C.E)$=0.58.

$\gamma(D,B)$=0.4.

$\vec{\dot{\underset{\sim}{\mathcal{R}}}}$	A	B	C	D	E
A	0	0,664	0,2	0,44	0,664
B	0,664	0	0,58	0,4	0
C	0,2	0,58	0	0,3	0,58
D	0,44	0,4	0,3	0	0,4
E	0,664	0	0,58	0,4	0

Fig 25.7

$\vec{\dot{\underset{\sim}{\mathcal{R}}}}$	0	1	2	3	4	5	6	7	8	...
0	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	...
1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	...
2	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	...
3	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	...
4	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	...
5	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	...
6	$\frac{6}{7}$	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{2}{3}$	...
7	$\frac{7}{8}$	$\frac{6}{7}$	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	...
8	$\frac{8}{9}$	$\frac{7}{8}$	$\frac{6}{7}$	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig 25.8

Example 2. We take up again the example of Figure 26.5. A max-product composition shows immediately that

(25.25) $$\hat{\dot{\underline{\mathcal{R}}}} = \underline{\mathcal{R}}$$

The relation $\vec{\dot{\underline{\hat{\mathcal{R}}}}}$ is given in Figure 26.8.

One sees that

$$\gamma(n_1, n_2) = \frac{|n_2 - n_1|}{|n_2 - n_1| + 1} \quad (25.26)$$

and that, as a consequence,

$$\lim_{|n_2-n_1|\to\infty} \gamma(n_1, n_2) = 1 \quad (25.27)$$

Remark. It appears that $\gamma(x, y)$ gives a better practical idea of distance than $d(x, y)$; this may be very important for all concerned with problems of resemblance, hence the interest that we have shown in the min-sum distance. But, as we shall go on to see in Figure 26.10 of the next section, decomposition into ordinary partial graphs is no longer possible.

Theorem I. Let o be a resemblance relation. Then one always has

$$\vec{\underline{\hat{\mathcal{R}}}} \subset \vec{\dot{\underline{\hat{\mathcal{R}}}}} \quad (25.28)$$

that is

$$\forall(x, y): d(x, y) \leq \gamma(x, y) \quad (25.29)$$

Proof. On account of max-min transitivity one has

$$\mu_{\underline{\mathcal{R}}}(x, z) \geq \vee_y \left[\mu_{\underline{\mathcal{R}}}(x, y) \wedge \mu_{\underline{\mathcal{R}}}(y, z)\right] \quad (25.20)$$

From max-product transitivity one has

$$\mu_{\underline{\mathcal{R}}}(x, z) \geq \vee_y \left[\mu_{\underline{\mathcal{R}}}(x, y) \, . \, \mu_{\underline{\mathcal{R}}}(y, z)\right] \quad (25.31)$$

But, according to (18.18),

$$\mu_{\underline{\mathcal{R}}}(x, y) \wedge \mu_{\underline{\mathcal{R}}}(y, z) \geq \mu_{\underline{\mathcal{R}}}(x, y) \, . \, \mu_{\underline{\mathcal{R}}}(y, z) \quad (25.32)$$

which implies

$$\underset{max-min}{\vee_y \left[\mu_{\underline{\mathcal{R}}}(x, y) \wedge \mu_{\underline{\mathcal{R}}}(y, z)\right]} \geq \underset{max-product}{\vee_y \left[\mu_{\underline{\mathcal{R}}}(x, y) \, . \, \mu_{\underline{\mathcal{R}}}(y, z)\right]} \quad (25.33)$$

that is

$$\underline{\mathcal{R}} \, . \, \underline{\mathcal{R}} \subset \underline{\mathcal{R}} \circ \underline{\mathcal{R}} \quad (25.34)$$

where, we recall, indicates max-product composition and max-min composition.

From here

(25.35) $$\dot{\hat{\underline{\mathcal{R}}}} \subset \hat{\underline{\mathcal{R}}}$$

And thus

(25.36) $$\vec{\hat{\underline{\mathcal{R}}}} \subset \vec{\dot{\hat{\underline{\mathcal{R}}}}}$$

Dissemblance relation. A relation $\underline{\mathcal{R}}$ such that

(25.37) 1) $\forall (x, x) \in E \times E \colon \mu_{\underline{\mathcal{R}}}(x, x) = 0$ antireflexivity

(25.38) 2) $\forall (x, y) \in E \times E \colon \mu_{\underline{\mathcal{R}}}(x, y) = \mu_{\underline{\mathcal{R}}}(x, y)$ symmetry

is called a *dissemblance relation*, Figure 25.9 gives an example.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	0	0,3	0,9	1	0,2
B	0,3	0	0,4	0,1	0
C	0,9	0,4	0	0,8	0,1
D	1	0,1	0,8	0	1
E	0,2	0	0,1	1	0

Fig. 25.9

We consider some evident properties, If is a resemblance relation, of is a dissemblance relation, and vice versa.

Theorem II. If $\hat{\underline{\mathcal{R}}}$ is the max-min transitive closure account of the resemblance relation,$\underline{\mathcal{R}}$ then$\vec{\hat{\underline{\mathcal{R}}}}$ is the min-max transitive closure of the corresponding dissemblance relation.

Proof. The max-min transitive closure is expressed by (17.8) and (17.3), thus

(25.39) $$\hat{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \underline{\mathcal{R}}^3 \cup \dots$$

and

(25.40) $$\mu_{\underline{\mathcal{R}} \circ \underline{\mathcal{R}}}(x, z) = \vee_y \left[\mu_{\underline{\mathcal{R}}}(x. y) \wedge \mu_{\underline{\mathcal{R}}}(y, z)\right]$$

The min-max transitive closure will then be expressed by†

(25.41) $$\breve{\underline{\mathcal{R}}} = \underline{\mathcal{R}} \cap \left(\underline{\mathcal{R}} \circ \underline{\mathcal{R}}\right) \cap \left(\underline{\mathcal{R}} \circ \underline{\mathcal{R}} \circ \underline{\mathcal{R}}\right) \cap \dots (')$$

and

(25.42) $$\mu_{\underline{\mathcal{R}} \circ \underline{\mathcal{R}}}(x, z) = \wedge_y \left[\mu_{\underline{\mathcal{R}}}(x. y) \vee \mu_{\underline{\mathcal{R}}}(y, z)\right]$$

Let $\underline{\mathcal{R}}$ be a resemblance relation; $\hat{\underline{\mathcal{R}}}$ is a similitude relation; $\vec{\underline{\mathcal{R}}}$ is a dissemblance relation and $\check{\vec{\underline{\mathcal{R}}}}$is a dissimilitude relation. We show that

(25.43) $$\vec{\hat{\underline{\mathcal{R}}}} = \check{\vec{\underline{\mathcal{R}}}}$$

We have already shown in (25.4)-(25.8) that if $\underline{\mathcal{R}}$is max-min transitive, then$\vec{\underline{\mathcal{R}}}$ is min-max transitive,

†One may denote$\underline{\mathcal{R}} \circ \underline{\mathcal{R}} = \underline{\mathcal{R}}^2$, if there is no danger of confusion with the min operation and$\underline{\mathcal{R}} \circ \underline{\mathcal{R}} \circ \ldots \circ \underline{\mathcal{R}} = \underline{\mathcal{R}}^n$

We show now that

(25.44) $$\underset{max-min}{\overline{\underline{\mathcal{R}} \circ \underline{\mathcal{R}}}} = \underset{min-max}{\overline{\underline{\mathcal{R}}} \circ \overline{\underline{\mathcal{R}}}}$$

In order to verify this, one proceeds as we did in (25.4)-(25.8):

(25.45) $$\mu_{\underline{\mathcal{R}} \circ \underline{\mathcal{R}}}(x, z) = \vee_y \left[\mu_{\underline{\mathcal{R}}}(x, y) \wedge \mu_{\underline{\mathcal{R}}}(y, z)\right]$$

(25.46) $$\mu_{\overline{\underline{\mathcal{R}} \circ \underline{\mathcal{R}}}}(x, z) = 1 - \mu_{\underline{\mathcal{R}} \circ \underline{\mathcal{R}}}(x, z)$$

$$= 1 - \vee_y \left[\mu_{\underline{\mathcal{R}}}(x, y) \wedge \mu_{\underline{\mathcal{R}}}(y, z)\right]$$

$$= \wedge_y \left[\mu_{\overline{\underline{\mathcal{R}}}}(x, y) \vee \mu_{\overline{\underline{\mathcal{R}}}}(y, z)\right]$$

$$= \mu_{\overline{\underline{\mathcal{R}}} \circ \overline{\underline{\mathcal{R}}}}(x, z)$$

This proves (25.44).

Now we write

(25.47) $$\overline{\hat{\underline{\mathcal{R}}}} = \overline{\underline{\mathcal{R}} \cup \underline{\mathcal{R}}^2 \cup \underline{\mathcal{R}}^3 \cup \ldots}$$

$$= \overline{\underline{\mathcal{R}} \cup \left(\underline{\mathcal{R}} \circ \underline{\mathcal{R}}\right) \cup \left(\underline{\mathcal{R}} \circ \underline{\mathcal{R}} \circ \underline{\mathcal{R}}\right) \cup \ldots}$$

$$= \overline{\underline{\mathcal{R}}} \cap \overline{\underline{\mathcal{R}} \circ \underline{\mathcal{R}}} \cap \overline{\underline{\mathcal{R}} \circ \underline{\mathcal{R}} \circ \underline{\mathcal{R}}} \cap \ldots$$

(applying De Morgan's theorem)

$$= \underline{\overline{\mathcal{R}}} \cap \underline{\overline{\mathcal{R}}} \circ \underline{\overline{\mathcal{R}}} \cap \underline{\overline{\mathcal{R}}} \circ \underline{\overline{\mathcal{R}}} \circ \underline{\overline{\mathcal{R}}} \cap \ldots$$

[according to (25.44)]

$$= \underline{\breve{\overline{\mathcal{R}}}}.$$

We shall see an example. We take again the resemblance relation given by Figure 25.1, whose corresponding similitude relation has been given in Figure 25.2 and whose matrix of distances in Figure 25.3. We meet these relations again in the calculations that end in $\underline{\breve{\overline{\mathcal{R}}}}$ in Figures 25.10d-h.

$\underline{\mathcal{R}}$	A	B	C	D	E
A	1	0,1	0,8	0,2	0,3
B	0,1	1	0	0,3	1
C	0,8	0	1	0,7	0
D	0,2	0,3	0,7	1	0,6
E	0,3	1	0	0,6	1

(a)

$\underline{\hat{\mathcal{R}}}$	A	B	C	D	E
A	1	0,6	0,8	0,7	0,6
B	0,6	1	0,6	0,6	1
C	0,8	0,6	1	0,7	0,6
D	0,7	0,6	0,7	1	0,6
E	0,6	1	0,6	0,6	1

(b)

$\underline{\overline{\hat{\mathcal{R}}}}$	A	B	C	D	E
A	0	0,4	0,2	0,3	0,4
B	0,4	0	0,4	0,4	0
C	0,2	0,4	0	0,3	0,4
D	0,3	0,4	0,3	0	0,4
E	0,4	0	0,4	0,4	0

(c)

$\underline{\overline{\mathcal{R}}}$	A	B	C	D	E
A	0	0,9	0,2	0,8	0,7
B	0,9	0	1	0,7	0
C	0,2	1	0	0,3	1
D	0,8	0,7	0,3	0	0,4
E	0,7	0	1	0,4	0

(d)

$\underline{\overline{\mathcal{R}}} \circ \underline{\overline{\mathcal{R}}}$	A	B	C	D	E
A	0	0,7	0,2	0,3	0,7
B	0,7	0	0,7	0,4	0
C	0,2	0,7	0	0,3	0,4
D	0,3	0,4	0,3	0	0,4
E	0,7	0	0,4	0,4	0

$\underline{\overline{\mathcal{R}}} \circ \underline{\overline{\mathcal{R}}}$ through min–max composition

(e)

$\underline{\overline{\mathcal{R}}}^3$	A	B	C	D	E
A	0	0,7	0,2	0,3	0,4
B	0,7	0	0,4	0,4	0
C	0,2	0,4	0	0,3	0,4
D	0,3	0,4	0,3	0	0,4
E	0,4	0	0,4	0,4	0

min–max composition

(f)

$\underline{\overline{\mathcal{R}}}^4$	A	B	C	D	E
A	0	0,4	0,2	0,3	0,4
B	0,4	0	0,4	0,4	0
C	0,2	0,4	0	0,3	0,4
D	0,3	0,4	0,3	0	0,4
E	0,4	0	0,4	0,4	0

min–max composition

(g)

$\underline{\breve{\overline{\mathcal{R}}}}$	A	B	C	D	E
A	0	0,4	0,2	0,3	0,4
B	0,4	0	0,4	0,4	0
C	0,2	0,4	0	0,3	0,4
D	0,3	0,4	0,3	0	0,4
E	0,4	0	0,4	0,4	0

intersection of min–max compositions

(h)

Fig. 25.10

Theorem II.may be extended to the case of any relation, without imposing that it be a resemblance relation; the proof remains valid. Thus we may announce a more general theorem.

Theorem III.† Let $\hat{\underline{\mathcal{R}}}$be the max-min transitive closure of any fuzzy relation $\underline{\mathcal{R}} \subset E \times E$ whatever, and let$\breve{\overline{\underline{\mathcal{R}}}}$ be the min-max transitive closure of$\overline{\underline{\mathcal{R}}}$. Then

(25.48) $$\overline{\hat{\underline{\mathcal{R}}}} = \breve{\overline{\underline{\mathcal{R}}}}$$

This may also be expressed by writing: One may permute the order of the operations$\wedge$and$-$, but $\wedge$becomes$\vee$ (or vice versa) in the permutation.

†We might have introduced this theorem earlier, in Section 17, hut with a didactic aim (not to overload any section, operating progressively), we report this useful and important theorem in Section 25, where we have a true need for the notion of distance.

With respect to these, the reader may seek other interesting properties concerning the max-min and min-max transitive closures, which one may characterize as duals without being the object of reproach for using that word.

27, VARIOUS PROPERTIES CONCERNING SIMILITUDE AND RESEMBLANCE

Theorem of decomposition for a similitude relation. Let$\underline{\mathcal{R}}$ be a similitude relation in $E \times E$. Then$\underline{\mathcal{R}}$ may be decomposed in the form

(27.1) $$\underline{\mathcal{R}} = \vee_{\alpha}\, \alpha . \mathcal{R}_{\alpha} \qquad 0 < \alpha < 1$$
$$with\, \alpha_1 > \alpha_2 \Rightarrow \mathcal{R}_2 \supset \mathcal{R}_1$$

where the $\mathcal{R}_{\alpha}$, are equivalence relations in the sense of ordinary set theory and $\alpha\mathcal{R}_{\alpha}$ indicates that all the elements of the ordinary relation $\mathcal{R}_{\alpha}$ are multiplied by α.

Proof. First, $\mu_{\underline{\mathcal{R}}}(x,x) = 1$; it follows that $(x,x) \in \mathcal{R}_{\alpha}$, for $\alpha \in [0,1]$; and thus $\mathcal{R}_{\alpha}$ has the property of reflexivity.

Then, letting $(x,y) \in \mathcal{R}_{\alpha}, \alpha \in [0,1]$, this implies that $\mu_{\underline{\mathcal{R}}}(x,y) \geq \alpha$ and$\mu_{\underline{\mathcal{R}}}(y,z) \geq \alpha$, by the symmetry of $\underline{\mathcal{R}}. \mu_{\underline{\mathcal{R}}}(y,x) \geq \alpha$. Then,$\mathcal{R}_{\alpha}$ has the property of symmetry.

Finally, for all $\alpha \in [0,1]$, suppose that $(x,y) \in \mathcal{R}_{\alpha}$, and $(y,z) \in \mathcal{R}_{\alpha}$; then $\mu_{\underline{\mathcal{R}}}(x,y) \geq \alpha$and $\mu_{\underline{\mathcal{R}}}(y,z) \geq \alpha$, then by transitivity, $\mu_{\underline{\mathcal{R}}}(x,z) \geq \alpha$ and also $\mathcal{R}_{\alpha}$ is transitive,

Then, $\mathcal{R}_{\alpha}$, being reflexive, symmetric, and transitive, is an equivalence relation.

The converse theorem is equally true.

Converse.$\mathcal{R}_1$, is nonempty, $(x,x) \in \mathcal{R}_1$, and also

(27.2) $$\mu_{\underline{\mathcal{R}}}(x,x) = 1, \quad \forall x \in E,$$

then$\underline{\mathcal{R}}$ is a reflexive fuzzy relation.

On the other hand, referring to (13.31), one may write

(27.3) $$\forall (x,y) \in E \times E: \mu_{\underline{\mathcal{R}}}(x,y) = \vee_{\alpha}\, \alpha . \mu_{\mathcal{R}_{\alpha}}(x,y)$$

It is evident that the symmetry of each $\mathcal{R}_{\alpha}$, implies the symmetry of $\underline{\mathcal{R}}$.

Finally, let

(27.4) $$\mu_{\underline{\mathcal{R}}}(x,y) = \alpha, \text{ and } \mu_{\underline{\mathcal{R}}}(y,z) = \beta;$$

Then

(27.5) $$(x,y) \in \mathcal{R}_{\alpha \wedge \beta}, \text{ and} (y,z) \in \mathcal{R}_{\alpha \wedge \beta}$$

As a consequence

(27.6) $$(x,z) \in \mathcal{R}_{\alpha \wedge \beta}$$

because$\mathcal{R}_{\alpha > \beta}$ is transitive.

It follows that

(27.7) $$\forall x,y,z \in E: \mu_{\underline{\mathcal{R}}}(x,z) \geq \alpha \wedge \beta$$

and also

(27.8) $$\mu_{\underline{\mathcal{R}}}(x,z) \geq \vee_{y} \left(\mu_{\underline{\mathcal{R}}}(x,y) \wedge \mu_{\underline{\mathcal{R}}}(y,z)\right)$$

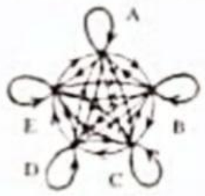
This with (27.2) and (27,3) proves the transitivity of $\underline{\mathcal{R}}$.

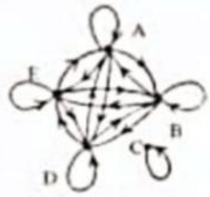
This converse allows the synthesis of similitude relations, as the direct theorem permits analysis,

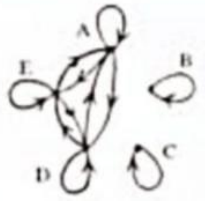
Interesting remark. It follows from this theorem that the ordinary relation closest to a similitude relation is an equivalence relation. This one may see immediately by considering what represents $\mathcal{R}_{\alpha}$, when $\alpha > 0{,}5$,

Examples. We now see the analysis of the relation given in Figure 20.1. The decomposition has been presented in Figure 27.1.

$$\begin{array}{c|ccccc} \mathcal{Q} & A & B & C & D & E \\ \hline A & 1 & 0,8 & 0,7 & 1 & 0,9 \\ B & 0,8 & 1 & 0,7 & 0,8 & 0,8 \\ C & 0,7 & 0,7 & 1 & 0,7 & 0,7 \\ D & 1 & 0,8 & 0,7 & 1 & 0,9 \\ E & 0,9 & 0,8 & 0,7 & 0,9 & 1 \end{array} = \vee \left((0,7) \cdot \overset{\mathcal{R}_{0,7}}{\begin{bmatrix} 1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1 \end{bmatrix}},\ (0,8) \cdot \overset{\mathcal{R}_{0,8}}{\begin{bmatrix} 1&1&0&1&1\\1&1&0&1&1\\0&0&1&0&0\\1&1&0&1&1\\1&1&0&1&1 \end{bmatrix}},\ (0,9) \cdot \overset{\mathcal{R}_{0,9}}{\begin{bmatrix} 1&0&0&1&1\\0&1&0&0&0\\0&0&1&0&0\\1&0&0&1&1\\1&0&0&1&1 \end{bmatrix}},\ (1) \cdot \overset{\mathcal{R}_{1}}{\begin{bmatrix} 1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\1&0&0&1&0\\0&0&0&0&1 \end{bmatrix}} \right)$$







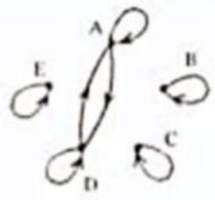


Fig. 27.1

Next we consider an example of synthesis, Let the four equivalence relations be successively included in one another (Figure 27.2):

$$\begin{array}{c|cccc} \mathcal{R}_{0,7} & A & B & C & D \\ \hline A & 1 & 1 & 1 & 1 \\ B & 1 & 1 & 1 & 1 \\ C & 1 & 1 & 1 & 1 \\ D & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cccc} \mathcal{R}_{0,8} & A & B & C & D \\ \hline A & 1 & 1 & 0 & 0 \\ B & 1 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & 1 \\ D & 0 & 0 & 1 & 1 \end{array} \qquad \begin{array}{c|cccc} \mathcal{R}_{0,9} & A & B & C & D \\ \hline A & 1 & 1 & 0 & 0 \\ B & 1 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{c|cccc} \mathcal{R}_{1} & A & B & C & D \\ \hline A & 1 & 0 & 0 & 0 \\ B & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \end{array}$$

(a) (b) (c) (d)

Fig. 27.2

One then has

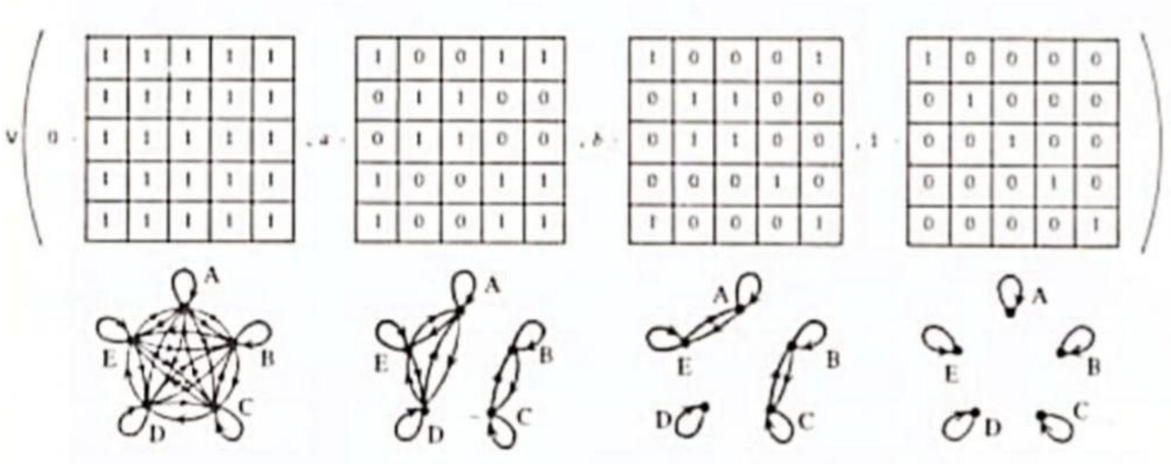
(27.9) $$\underline{\mathcal{R}} = \vee\left(0{,}2\,.\,\mathcal{R}_{0,2}\,,\,0{,}6\,.\,\mathcal{R}_{0,6}\,,\,0{,}8\,.\,\mathcal{R}_{0,8}\,,\,1.\,\mathcal{R}_{1}\right)$$

The result is shown in Figure 27.3.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D
A	1	0,8	0,2	0,2
B	0,8	1	0,2	0,2
C	0,2	0,2	1	0,6
D	0,2	0,2	0,6	1

Fig.27.3

Another example is shown in Figure 27.4, where we have supposed that a and $b \in [0,1]$with $a < b$.



	A	B	C	D	E
A	1	0	0	a	b
B	0	1	b	0	0
C	0	b	1	0	0
D	a	0	0	1	a
E	b	0	0	a	1

Fig. 27.4

Transitive graphs of distances. It is interesting to present for each similitude relation the transitive graphs corresponding to the min-max distances, some examples serve to demonstrate the interest.

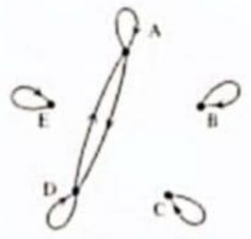
Example 1. Figure 27.5 gives an example of a dissimilitude relation. In Figure 27.6 we have represented the transitive graphs corresponding to various distances.

$\underset{\sim}{\mathcal{R}}$	A	B	C	D	E
A	0	0,2	0,8	0	0,1
B	0,2	0	0,8	0,2	0,2
C	0,8	0,8	0	0,8	0,8
D	0	0,2	0,8	0	0,1
E	0,1	0,2	0,8	0,1	0

Fig 27.5

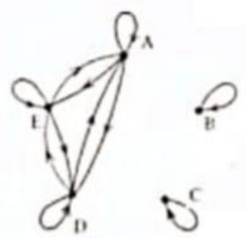
distances equal to 0:

	A	B	C	D	E
A	1	0	0	1	0
B	0	1	0	0	0
C	0	0	1	0	0
D	1	0	0	1	0
E	0	0	0	0	1



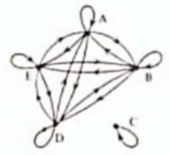
distances less than or equal to 0,1:

	A	B	C	D	E
A	1	0	0	1	1
B	0	1	0	0	0
C	0	0	1	0	0
D	1	0	0	1	1
E	1	0	0	1	1



distances less than or equal to 0,2:

	A	B	C	D	E
A	1	1	0	1	1
B	1	1	0	1	1
C	0	0	1	0	0
D	1	1	0	1	1
E	1	1	0	1	1



distances less than or equal to 0,8:

	A	B	C	D	E
A	1	1	1	1	1
B	1	1	1	1	1
C	1	1	1	1	1
D	1	1	1	1	1
E	1	1	1	1	1

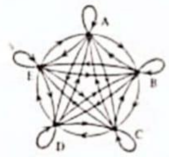


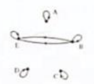
Fig. 27.6 Transitive graphs of distance

Example 2. (Figures 27.7 and 27.8). This example is relative to the transitive closure (Figure 26.2) of the resemblance relation (Figure 26.1). The decomposition obtained will be compared to that of the following example (Figures 27.9 and 27.10).

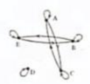
$\bar{\mathcal{R}}$	A	B	C	D	E
A	0	0,4	0,2	0,3	0,4
B	0,4	0	0,4	0,4	0
C	0,2	0,4	0	0,3	0,4
D	0,3	0,4	0,3	0	0,4
E	0,4	0	0,4	0,4	0

Fig. 27.7

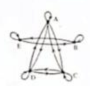
	A	B	C	D	E
A	1	0	0	0	0
B	0	1	0	0	1
C	0	0	1	0	0
D	0	0	0	1	0
E	0	1	0	0	1



	A	B	C	D	E
A	1	0	1	0	0
B	0	1	0	0	1
C	1	0	1	0	0
D	0	0	0	1	0
E	0	1	0	0	1



	A	B	C	D	E
A	1	0	1	1	0
B	0	1	0	0	1
C	1	0	1	1	0
D	1	0	1	1	0
E	0	1	0	0	1



	A	B	C	D	E
A	1	1	1	1	1
B	1	1	1	1	1
C	1	1	1	1	1
D	1	1	1	1	1
E	1	1	1	1	1

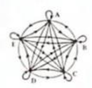
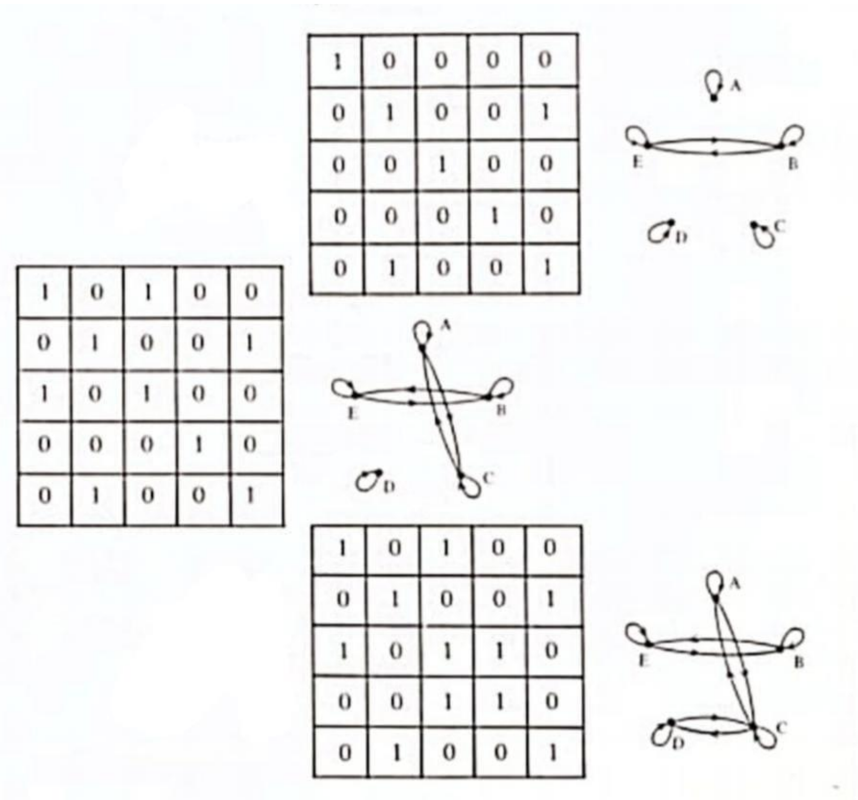


Fig. 27.8 Transitivity graphs of min-max distances

Example 3. (Figures 27,9 and 27.10). The max-product transitive closure of the resemblance relation of Figure 26,1 has been obtained in Figure 26.6. For this, we have drawn in Figure 26,7 the matrix of min-sum distances. The decomposition into ordinary graphs of distances that will not all be transitive appears in this example. It is an inconvenience to use the max-product transitive closure in a resemblance relation in comparison to the use of the max-min transitive closure.

$\hat{\underset{\sim}{\mathcal{R}}}$	A	B	C	D	E
A	0	0,664	0,2	0,44	0,664
B	0,664	0	0,58	0,4	0
C	0,2	0,58	0	0,3	0,58
D	0,44	0,4	0,3	0	0,4
E	0,664	0	0,58	0,4	0

Fig. 27.9

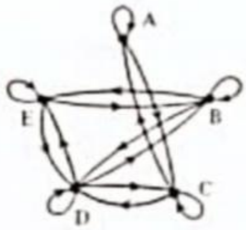


1	0	0	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	1	0
0	1	0	0	1

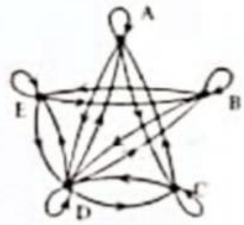
1	0	1	0	0
0	1	0	0	1
1	0	1	0	0
0	0	0	1	0
0	1	0	0	1

1	0	1	0	0
0	1	0	0	1
1	0	1	1	0
0	0	1	1	0
0	1	0	0	1

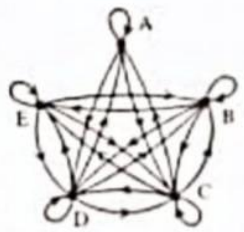
1	0	1	0	0
0	1	0	1	1
1	0	1	1	0
0	1	1	1	1
0	1	0	1	1



1	0	1	1	0
0	1	0	1	1
1	0	1	1	0
1	1	1	1	1
0	1	0	1	1



1	0	1	1	0
0	1	1	1	1
1	1	1	1	1
1	1	1	1	1
0	1	1	1	1



1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

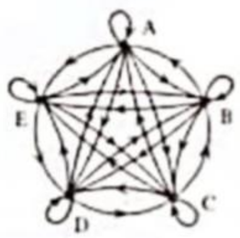


Fig. 27.10

Tree decomposition. A reader who examines Figure 27.1 is led to note that gradually as a takes the values 0,7, 0,8, 0,9, and 1, the partition of E into equivalence classes includes more and more parts. This decomposition has been carried out according to a tree scheme, which has been represented in Figure 27.11. An ordered scheme such as this is called tree decomposition.

Another example relative to Figure 27,4 is given in Figure 27.12.

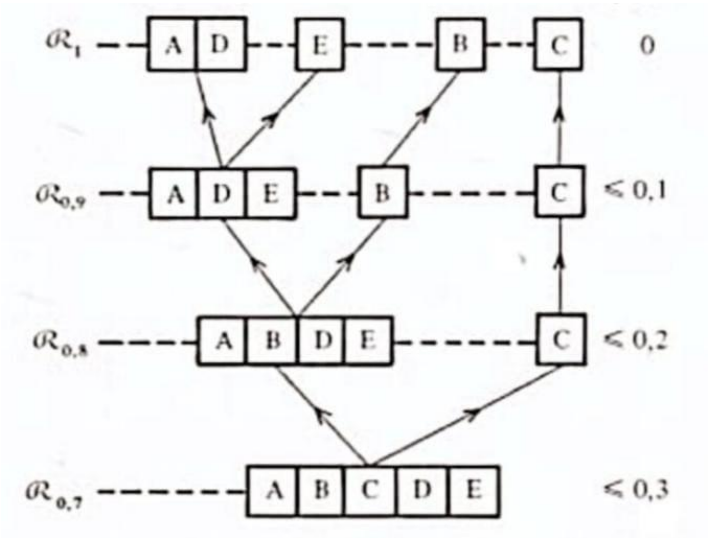


Fig. 27.11

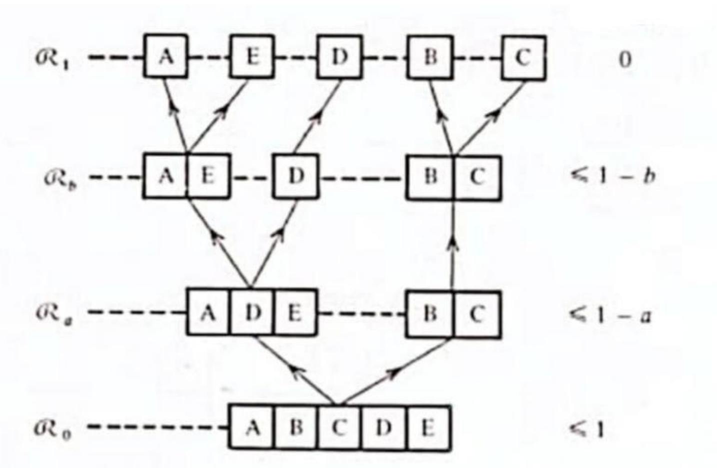


Fig. 27.12

One may verify that two elements x and y belonging to E would belong to the same class of level α if and only if

(27.10) $$\mu_{\mathcal{R}}(x, y) > \alpha$$

This decomposition tree reflects the structure of the similitude relation well, or if one prefers, the groupings of elements by their transitive distances from one another.

One may represent a tree in various manners. Using the notation of linguistics, one may write sequentially following the tree of Figure 27.11

(27.11) $$0{,}7\left(0{,}8\big(0{,}9(1\{A,D\},1\{E\}),0{,}9(1\{B\})\big),0{,}8\big(0{,}9(1\{C\})\big)\right)$$

Such a use of parentheses is not convenient.

One may also use the notion of a pile and represent the tree (27.11) with the sequence: 0,7 (ABCDE) 0,8 (ABDE) 0,9 (ADE) 1 (AE) 0,9 (ADE) 1 (D) 0,9 (ADE) 0,8 (ABDE) 0,9 (B) 1 (B) 0,9 (B) 0,8 (ABDE) 0,7 (ABCDE) 0,8 (C) 0,9 (C) 1 (C) 0,9 (C) 0,8 (C) 0,7 (ABCDE). This notation is associated with the scheme known as Polish notation,

It is easy to follow the sequence in Figure 27.11.

Selection of the transitively nearest messages. One may consider a fuzzy subset as a message that is fuzzy instead of being binary.

Consider an ordinary set F of fuzzy subsets A, belonging to the same reference set E:

(27.12) $$F = \{\underline{A}_1, \underline{A}_2, \dots, \underline{A}_n\}$$

We have in mind the determination of which fuzzy subsets or fuzzy messages are transitively nearest. We shall make precise a little later the inconveniences of the notion of transitivity that will be considered, the advantages are at once apparent.

We shall proceed as follows (and shall explain at the same time what is meant by transitively nearest):

(1) For each pair $(\underline{A}_i, \underline{A}_j), i, j = 1,2, \dots, n$, evaluate the relative generalized Hamming distance† $\delta(\underline{A}_i, \underline{A}_j)$; this gives a dissemblance relation $\precsim$.

(2) Take the min-max transitive closure [that defined by (26.41)]. The relation $\check{\precsim}$ obtained gives the min-max transitive distance:

(27.13) $$\check{\delta}(\underline{A}_i, \underline{A}_j)$$

(3) Then decompose $\check{\precsim}$ according to (27.1) and obtain the following ordinary subsets of F:

transitively nearest messages for which one has

(27.14) $$\check{\delta}(\underline{A}_i, \underline{A}_j) = 0$$

transitively nearest messages for which one has

(27.15) $$0 < \check{\delta}(\underline{A}_i, \underline{A}_j) = \alpha_1 < \alpha_2 < \cdots$$

transitively nearest messages for which one has

(27.16) $$0 < \alpha_1 < \check{\delta}(\underline{A}_i, \underline{A}_j) = \alpha_2 < \alpha_3 < \cdots$$

†Or relative euclideandistance $(\underline{A}_i, \underline{A}_j)$, this depending on the nature of the problem, or even someother notion of distance.

And so on.

(4) Construct the corresponding composition tree.

Example. Let E be a finite reference set with $card(E) = 7$ and consider six subsets or messages $\underline{A}_i, i = 1,2, \dots ,6$.

(27.17)

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
$\underset{\sim}{A}_1 =$	0,1	0,8	0,3	1	0,1	0	1	,
$\underset{\sim}{A}_2 =$	0,3	0,8	0,1	1	0	1	0,7	,
$\underset{\sim}{A}_3 =$	0,7	1	0	1	0,8	0,3	0,7	,
$\underset{\sim}{A}_4 =$	0,1	0,8	0,7	0	0,1	1	0,3	,
$\underset{\sim}{A}_5 =$	0,6	1	0	0,7	0,8	0	1	,
$\underset{\sim}{A}_6 =$	0	0,3	0,5	0,1	0,1	0,5	0,8	.

Then calculate the relative generalized Hamming distance:

(27.18) $$\delta(\underline{A}_i, \underline{A}_j) = \frac{d(\underline{A}_i, \underline{A}_j)}{7}$$

This gives the dissemblance relation $\lesssim$ (Figure 27,13a). One then calculates with the aid of (26.41) the min-max transitive closure$\lesssim$, which gives the transitive distances δ, (See Figures 27,14 and 27.15.)

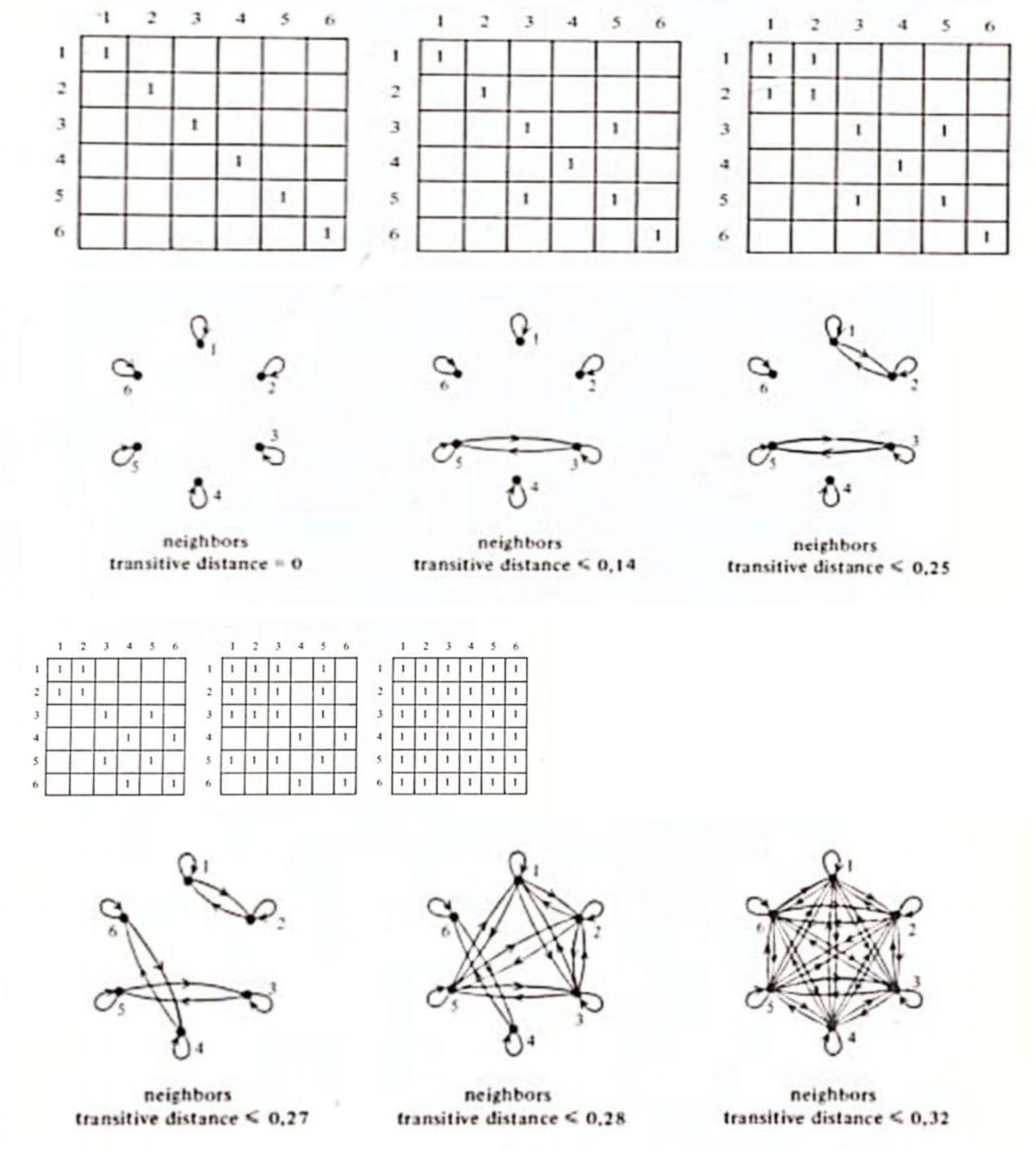
$\tilde{\delta}$	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	$\tilde{A}_5$	$\tilde{A}_6$
$\tilde{A}_1$	0	0,25	0,34	0,44	0,28	0,34
$\tilde{A}_2$	0,25	0	0,31	0,32	0,42	0,40
$\tilde{A}_3$	0,34	0,31	0	0,61	0,14	0,54
$\tilde{A}_4$	0,44	0,32	0,61	0	0,64	0,27
$\tilde{A}_5$	0,28	0,42	0,14	0,64	0	0,54
$\tilde{A}_6$	0,34	0,40	0,54	0,27	0,54	0

(a)

$\check{\tilde{\delta}}$	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	$\tilde{A}_5$	$\tilde{A}_6$
$\tilde{A}_1$	0	0,25	0,28	0,32	0,28	0,32
$\tilde{A}_2$	0,25	0	0,28	0,32	0,28	0,32
$\tilde{A}_3$	0,28	0,28	0	0,32	0,14	0,32
$\tilde{A}_4$	0,32	0,32	0,32	0	0,32	0,27
$\tilde{A}_5$	0,28	0,28	0,14	0,32	0	0,32
$\tilde{A}_6$	0,32	0,32	0,32	0,27	0,32	0

(b)

Fig. 27.13



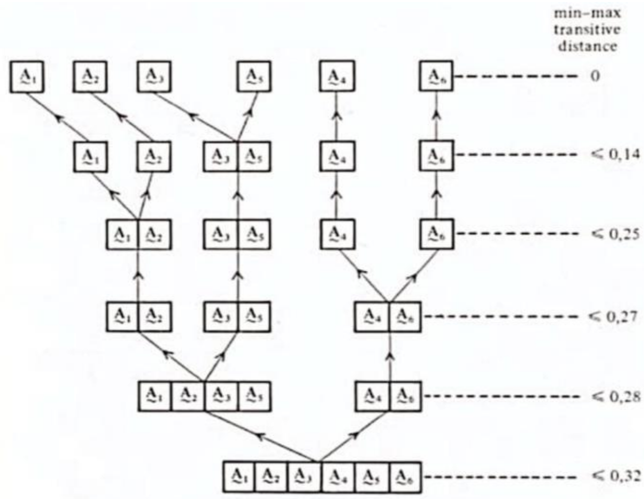


Fig. 27.15

Important remark on the subject of transitive distance. Depending on the nature of the problem being treated, the min-max transitive closure of a distance matrix may not be significant in its practical employmen., We consider an example. Consider the following four messages:

$$\underset{\sim}{A}_1 = \begin{array}{cc} x_1 & x_2 \\ \hline \boxed{0} & \boxed{0} \end{array}$$

$$\underset{\sim}{A}_2 = \begin{array}{cc} x_1 & x_2 \\ \hline \boxed{0} & \boxed{\tfrac{1}{2}} \end{array}$$

$$\underset{\sim}{A}_3 = \begin{array}{cc} x_1 & x_2 \\ \hline \boxed{\tfrac{1}{2}} & \boxed{0} \end{array}$$

$$\underset{\sim}{A}_4 = \begin{array}{cc} x_1 & x_2 \\ \hline \boxed{0} & \boxed{1} \end{array}$$

$\overline{\underset{\sim}{\mathcal{R}}}$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(0, 1)$
$(0, 0)$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$(0, \frac{1}{2})$	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{1}{4}$
$(\frac{1}{2}, 0)$	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{3}{4}$
$(0, 1)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	0

Fig. 27.16

$\check{\overline{\underset{\sim}{\mathcal{R}}}}$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(0, 1)$
$(0, 0)$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$(0, \frac{1}{2})$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$(\frac{1}{2}, 0)$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$
$(0, 1)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0

Fig. 27.17

The relative generalized Hamming distances for these messages are given in Figure 27.16, which then constitutes a dissemblance matrix $\overline{\underset{\sim}{\mathcal{R}}}$. In Figure 27.17 we have calculated the min-max closure of $\overline{\underset{\sim}{\mathcal{R}}}$, that is, $\check{\overline{\underset{\sim}{\mathcal{R}}}}$. One sees then that all these messages are transitively equidistant.

This conception of min-max transitive distance may seem to be unacceptable in numerous applications. But the relative generalized Hamming distance is transitive for the ordinary min-addition operation, that is,

$$\delta(x,z) \leq \underset{y}{\mathrm{MIN}}[\delta(x,y)+\delta(y,z)] \quad (27.19)$$

Since, this is a distance, that is,

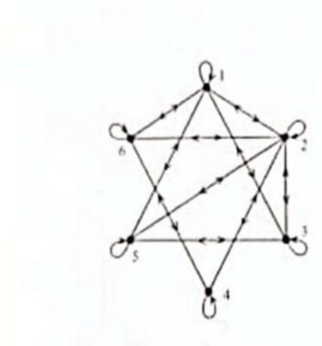
$$\forall y\colon \delta(x,z) \leq \delta(x,y)+\delta(y,z) \quad (27.20)$$

One comes to the same conclusions for relative euclidean distance.

Thus, any relation$\lesssim$ giving the relative generalized Hamming distance (or relative euclidean distance) is a relation that is its own ordinary min-addition transitive closure. Note that the member on the right-hand side of (27.19) may give a sum greater than 1. since it is an ordinary addition, but this constrains nothing since the member on the left, by construction, always belongs to $[0,1]$.

The decomposition by levels relative to values contained in the dissemblance relation will no longer give equivalence classes, but maximal subrelations, as we explain hereafter.

Ordinary min-addition disimilitude. Decomposition into maximal subrelations. The relation (27.19) may be considered as a dissimilitude relation, which we may call ordinary min-addition dissimilitude. As may be seen in the example given in Figure 27.19.



	1	2	3	4	5	6
1	1	1	1		1	1
2	1	1	1	1	1	1
3	1	1	1		1	
4		1		1		1
5	1	1	1		1	
6	1	1		1		1

	1	2	3	5
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
5	1	1	1	1

	1	2	6
1	1	1	1
2	1	1	1
6	1	1	1

	2	4	6
2	1	1	1
4	1	1	1
6	1	1	1

	1	2	3	5
1	0	0,25	0,34	0,28
2	0,25	0	0,31	0,42
3	0,34	0,31	0	0,14
5	0,28	0,42	0,14	0

	1	2	6
1	0	0,25	0,34
2	0,25	0	0,40
6	0,34	0,40	0

	2	4	6
2	0	0,32	0,40
4	0,32	0	0,27
6	0,40	0,27	0

Nondisjoint maximal similitude subrelations, $d \leq 0{,}42$

Fig. 27.18

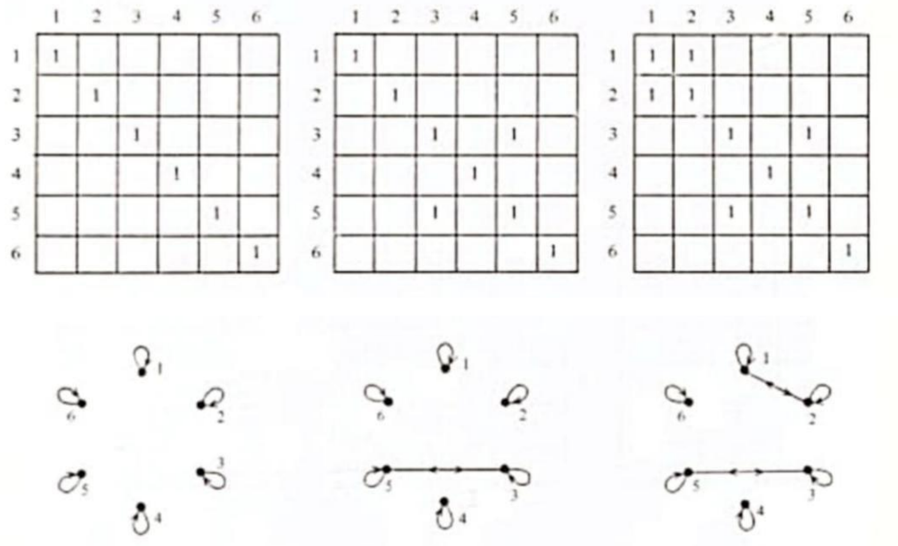
one does not obtain for a distance $d \leq k$ (k arbitrary) ordinary graphs whose subgraphs constitute equivalence classes, Sometimes one may use a less strong concept, which is rather interesting for various operations, that of maximal subrelations-which may be or may not be disjoint.

Take the case of Figure 27.19 and more particularly that of the ordinary symmetric graph corresponding to $d \leq 0{,}42$. In Figure 27.18 we have reproduced this ordinary graph and made evident three maximal subrelations or complete ordinary graphs, eachconstituting an equivalence relation. For each of these subrelations, the distance of each element to another is less than or equal to 0,42 and property (27.19) is verified. In general, such a decomposition may not be made without an appropriate algorithm, we give two of these in Appendix B, page 387.

Remark. Ordinary min-addition dissimilitude is not dual to that of max-product similitude; it is algebraic min-sum dissimilitude that corresponds in this duality see (26.33)].

We shall see a completely developed example where there appear maximal sub relations.

Example. We decompose the dissemblance relation (27.13a) (Figure 27.19)

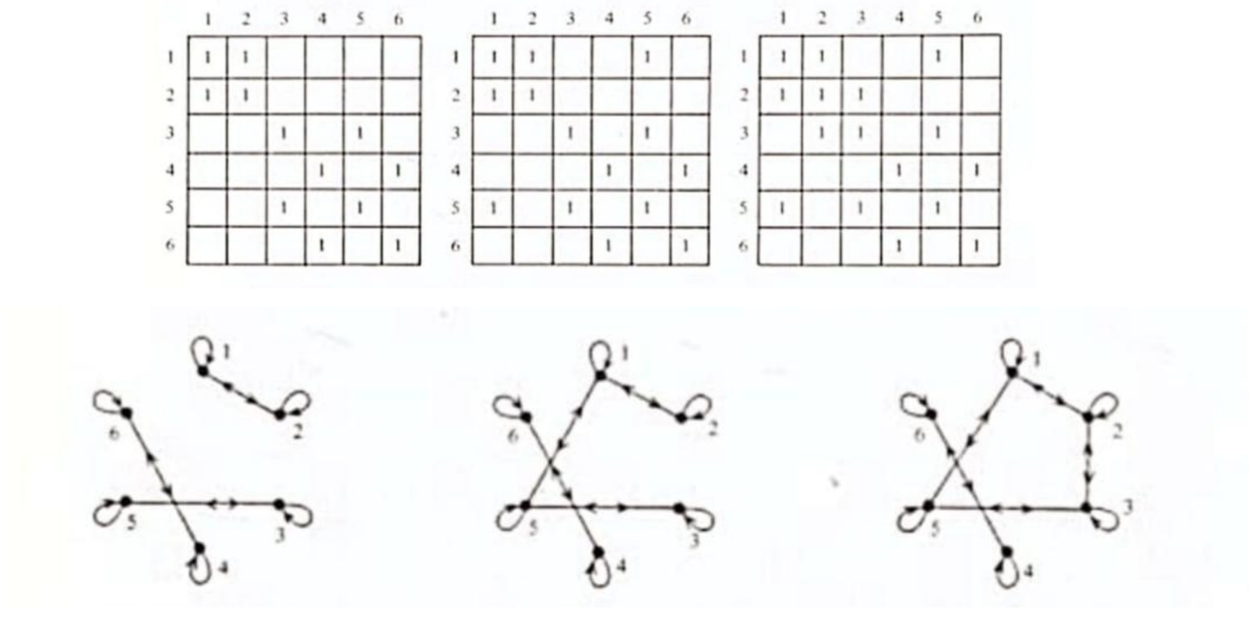


$\{1\},\{2\},\{3\},$ $\{4\},\{5\},\{6\}$ distance$=0$

$\{1\},\{2\},\{3,5\}$ $\{4\},\{6\}$ distance $<0{,}14$

$\{1,2\},\{3,5\}$ $\{4\},\{6\}$ distance$<0{,}25$

Fig. 27.19



$\{1,2\},\{3,5\},\{4,6\}$ distance$<0{,}27$

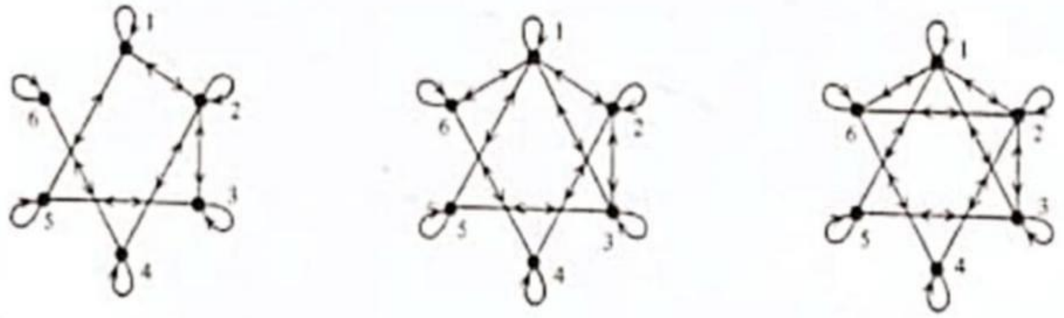
$\{1,2\},\{1,5\},\{3,5\}$ $\{4,6\}$ distance$<0{,}28$

$\{1,2\},\{1,5\},\{2,3\}$ $\{3,5\},\{4,6\}$ distance$<0{,}31$

	1	2	3	4	5	6
1	1	1			1	
2	1	1	1	1		
3		1	1		1	
4		1		1		1
5	1		1		1	
6				1		1

	1	2	3	4	5	6
1	1	1	1		1	1
2	1	1	1	1		
3	1	1	1		1	
4		1		1		1
5	1		1		1	
6	1			1		1

	1	2	3	4	5	6
1	1	1	1		1	1
2	1	1	1	1		1
3	1	1	1		1	
4		1		1		1
5	1		1		1	
6	1	1		1		1



{1,2}, {1,5}, {2,3}, {2,4} {3,5}, {4,6} distance< 0,32

{1,2,3}, {1,3,5} {1,6}, {2,4}, {4,6} distance< 0,34

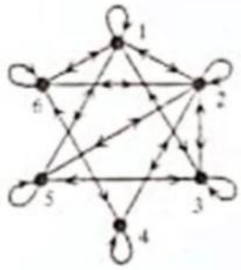
{1,2,3}, {1,2,6}, {1,3,5} {2,4,6} distance< 0,40

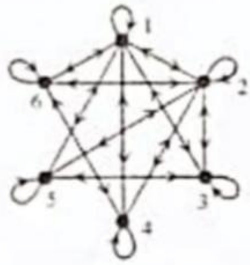
Fig. 27.19 (suite)

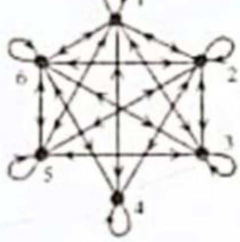
	1	2	3	4	5	6
1	1	1	1		1	1
2	1	1	1	1	1	1
3	1	1	1		1	
4		1		1		1
5	1	1	1		1	
6	1	1		1		1

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1		1	
4	1	1		1		1
5	1	1	1		1	
6	1	1		1		1

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1		1	1
4	1	1		1		1
5	1	1	1		1	1
6	1	1	1	1	1	1







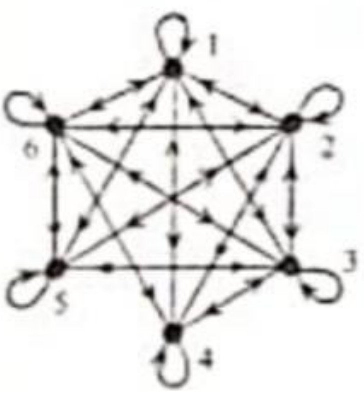
$\{1,2,3,5\},\{1,2,6\}$ $\{2,4,6\}$
distance$<$ 0,42

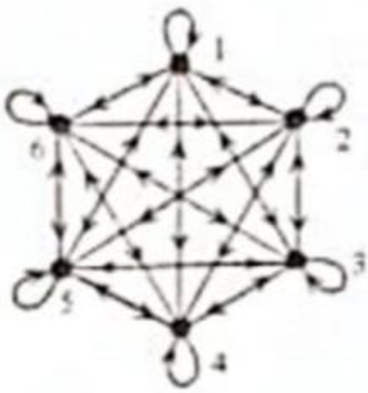
$\{1,2,4,6\},\{1,2,3,5\}$
distance$<$ 0,44

$\{1,2,3,5,6\},\{1,2,4,6\}$
distance$<$ 0,54

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1		1
5	1	1	1		1	1
6	1	1	1	1	1	1

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	1	1	1	1	1	1





$\{1,2,3,5,6\},\{1,2,3,4,6\}$
distance$<$ 0,61

$\{1,2,3,4,5,6\}$
distance$<$ 0,64

Fig. 27.19

Finally, one may also use the algebraic min-sum ($a \hat{+} b = a + b - ab$) transitivity to obtain the decomposition into maximal subrelations.

By comparing Figures 27.14 and 27.19, one may see the advantages and inconveniences of using min-max transitivity on the one hand and min-addition transitivity on the other. The first gives equivalence classes that are formed gradually depending on a, in contrast, interpretation is very debatable. The other gives only maximal subrelations that are not generally disjoint; but the interpretation is incontestable, particularly as concerns applications in the domain of classification of structures.

28. VARIOUS PROPERTIES OF FUZZY PERFECT ORDER RELATIONS

Decomposition theorem for a fuzzy perfect order relation. Let be a fuzzy perfect order relation in $E \times E$. The $\underline{\mathcal{R}}$may be decomposed in the form

(28.1) $$\underline{\mathcal{R}} = \vee_{\alpha}\, \alpha . \mathcal{R}_{\alpha}, \quad 0 < \alpha \leq 1$$

With $\alpha_1 \geq \alpha_2 \Rightarrow \mathcal{R}_{\alpha_1} \subset \mathcal{R}_{\alpha_2}$

where the $\mathcal{R}_{\alpha}$ are order relations in the sense of the theory of ordinary sets, and $\alpha . \mathcal{R}_{\alpha}$ expresses the product of all elements of $\mathcal{R}_{\alpha}$by the quantity α.

Proof. Reflexivity and transitivity of $\mathcal{R}_{\alpha}$ are proved as was (27.1) in Section 27. We shall see that this happens also for perfect antisymmetry according to (22.8).

In order to show the antisymmetry of $\mathcal{R}_{\alpha}$ we remark first that, since$\mathcal{R}_{\alpha}$ is reflexive, one may replace the definition

(28.2) $$\mu_{\mathcal{R}}(x, y) > 0 \Rightarrow \mu_{\mathcal{R}}(y, x) = 0$$

by

(28.3) $$(\mu_{\mathcal{R}}(x, y) > 0 \quad and \quad \mu_{\mathcal{R}}(y, x) = 0) \Rightarrow (x = y)$$

We shall reason by contradiction.

Suppose that $(x, y) \in \mathcal{R}_{\alpha}$and $(y, x) \in \mathcal{R}_{\alpha}$. Then $\mu_{\underline{\mathcal{R}}}(x, y) \geq \alpha$and $\mu_{\underline{\mathcal{R}}}(y, x) \geq \alpha$. Thus by the antisymmetryof $\underline{\mathcal{R}}$. $x = y$. Conversely, suppose $\mu_{\underline{\mathcal{R}}}(x, y) = \alpha > 0$and.$\mu_{\underline{\mathcal{R}}}(y, x) = \beta \geq 0$.Put $\gamma = \alpha \geq \beta$. Then $(x, y) \in \mathcal{R}_{\gamma}$and $(y, x) \in \mathcal{R}_{\gamma}$, and from the antisymmetry of $\mathcal{R}_{\gamma}$, it follows that $x = y$. But one may not have $x \neq y$ with these hypotheses.

Example 1. Figure 28.1 represents a decomposition of a fuzzy perfect order relation. To simplify reading of the results, we have omitted the zeros, Beneath each $\mathcal{R}_\alpha$ wehave placed a sketch representing the ordinary antisymmetric graph.

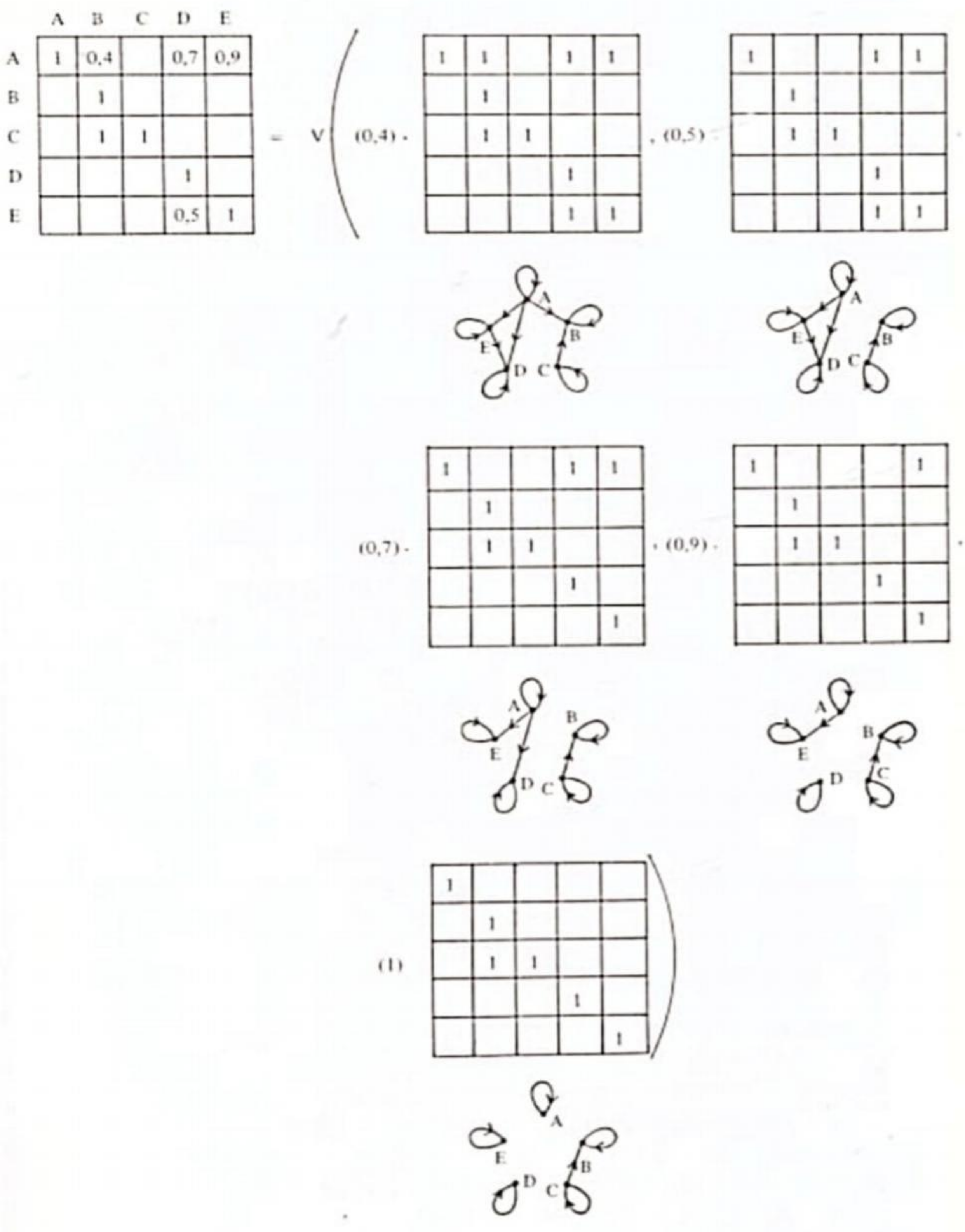


Fig. 28.1

Example 2. We see how to realize a synthesis of a perfect order relation (Figure 28.2).

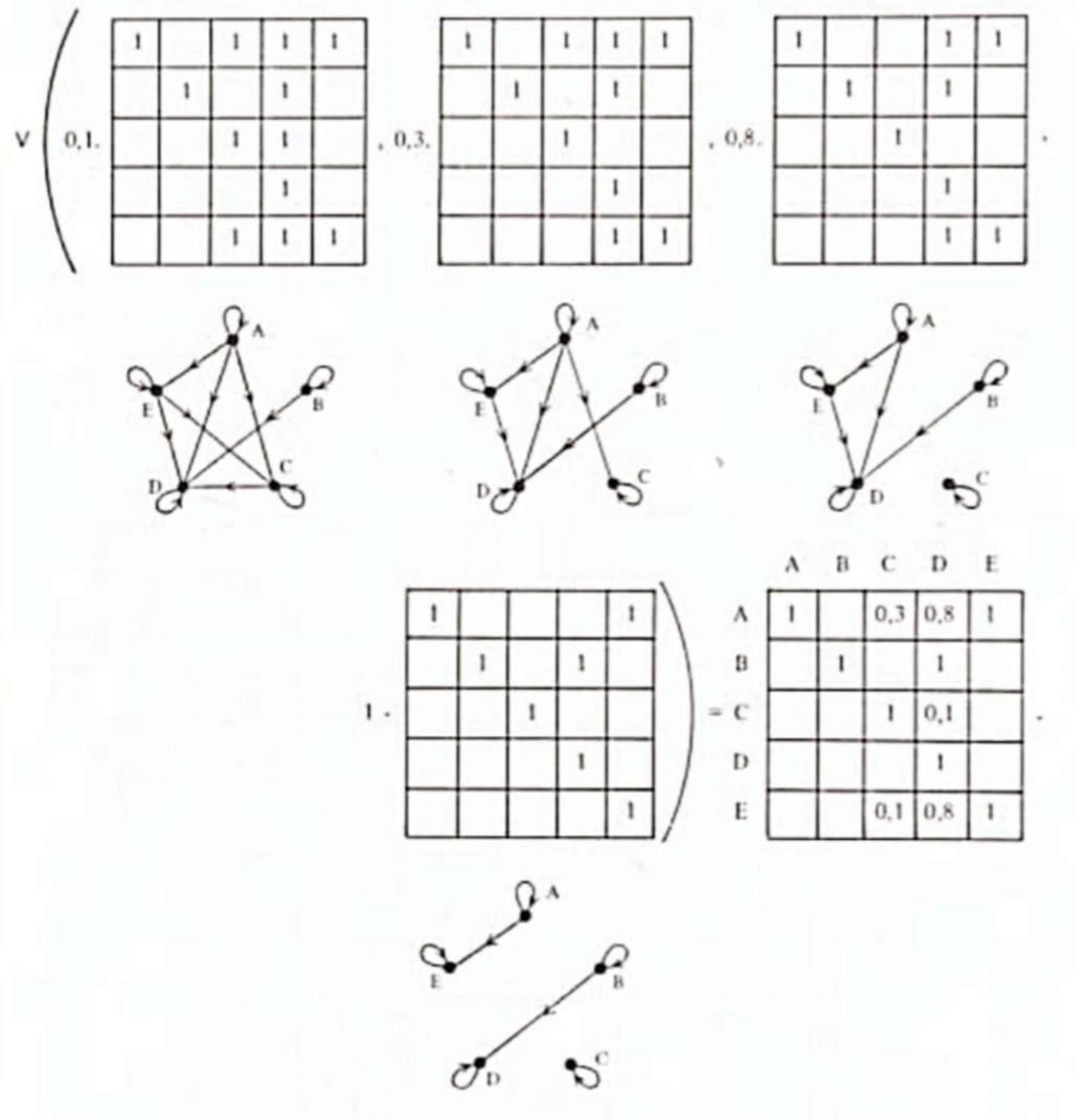


Fig. 28.2

Extension of the decomposition property to the case of a reducible preorder whose similitude classes are perfectly ordered. Properties (27.1) and (28.1) are combined whenever one considers a reducible preorder whose similitude classes constitute a perfect order.

Example. Figure 28.3 (pp. 163-164) gives an example of such a decomposition. In this figure the zeros are omitted to allow rapid examination. On the other hand, there are numerous elements and similitude classes for which the properties are readily apparent.

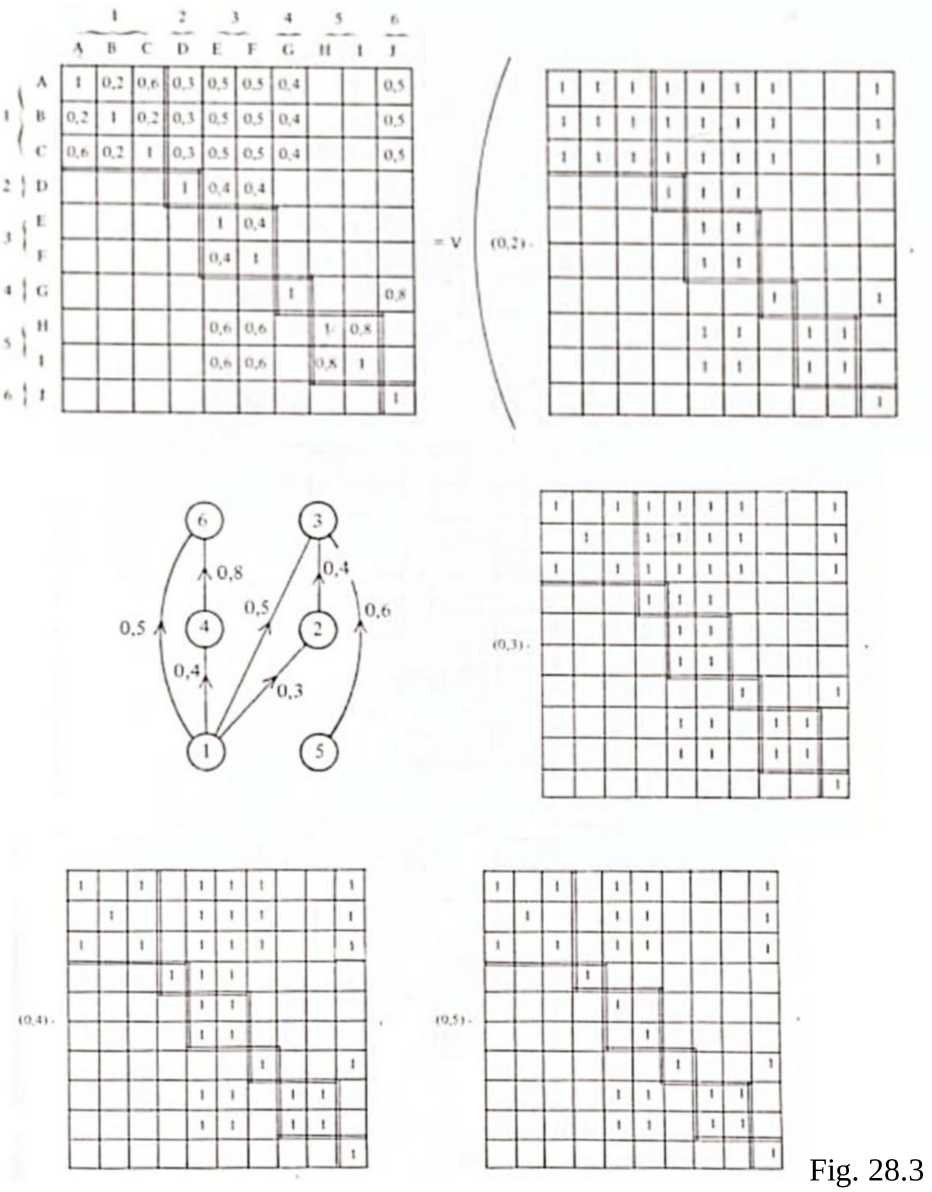


Fig. 28.3

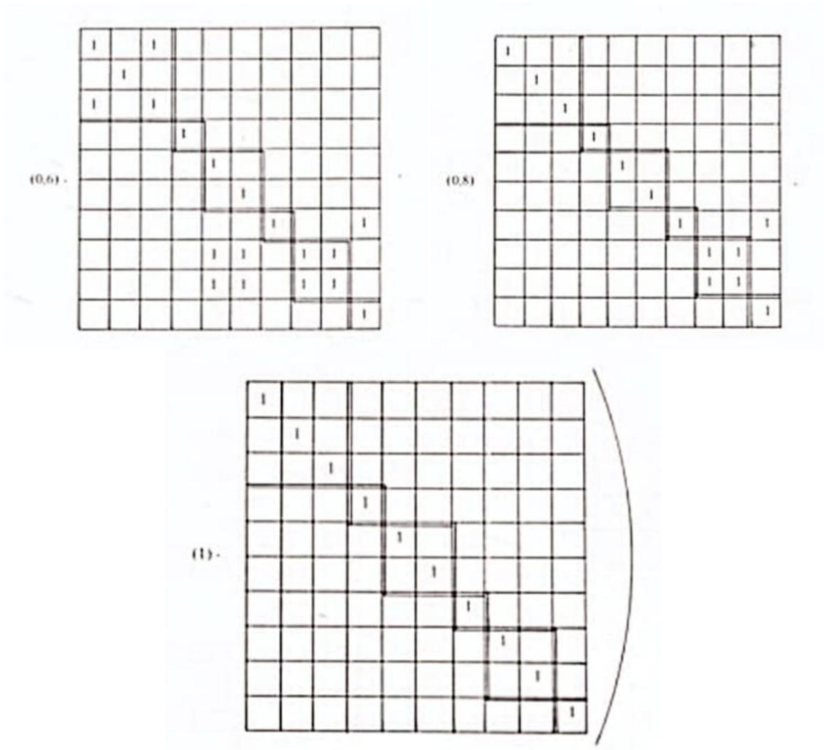
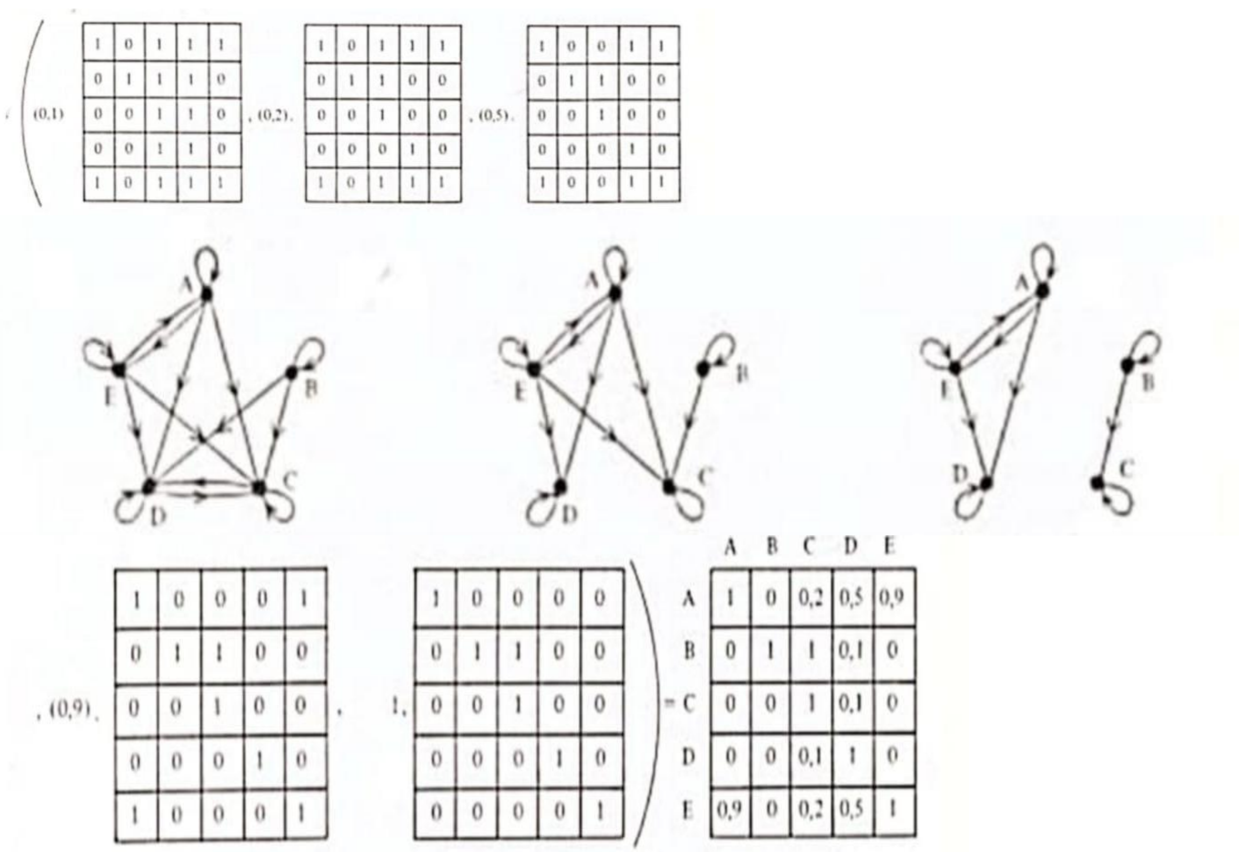


Fig. 28.3

Another example of synthesis. See Figure 28.4. Figure 28.5 illustrates the block-triangular form of the preorder.



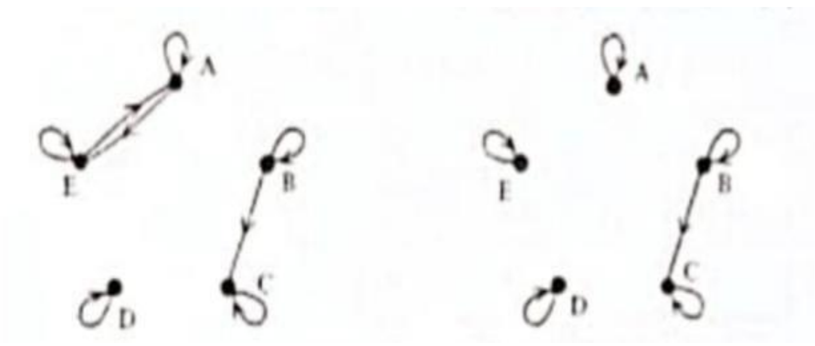
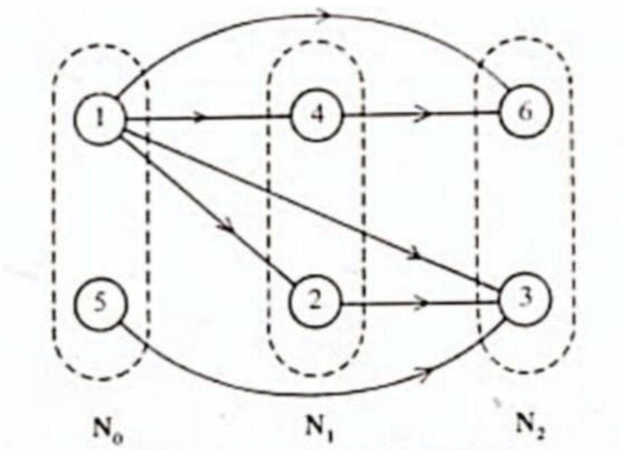


Fig. 28.4

$\underset{\sim}{\mathcal{R}}$	A	E	B	C	D
A	1	0,9	0	0,2	0,5
E	0,9	1	0	0,2	0,5
B	0	0	1	1	0,1
C	0	0	0	1	0,1
D	0	0	0	0,1	1

Fig. 28.5

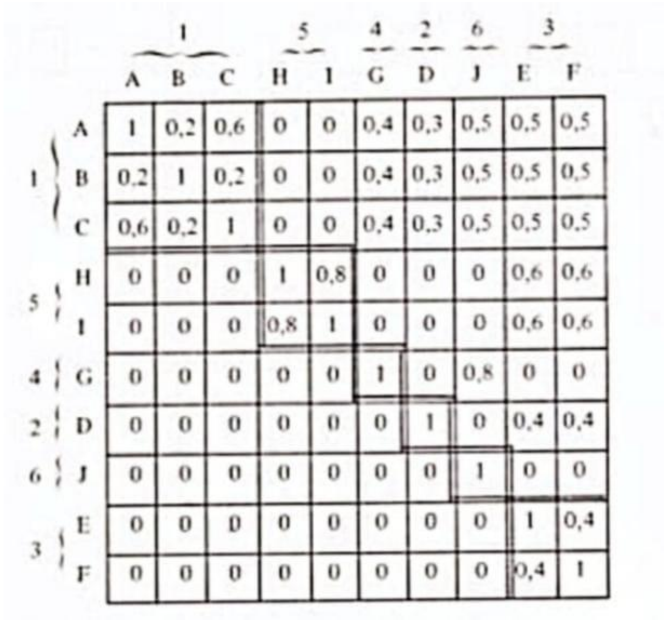
Perfect total order induced in a perfect partial order by the ordinal function (ease where E is finite).†Recalling what we have seen in Section 24, we take up again the example of Figure 28.3 and seek an ordinal function for the ordinary graph representing the order of the classes, consider the graph of Figure 28.6 in which appear three levels N_0, N_1, N_2.



In this figure a class C_i is represented by its index $i = 1,2,\dots,6$.

Fig. 28.6

These levels induce in the set of classes $\{C_1, C_2, \dots, C_6\}$ a (nonunique) total order such that, with respect to this order, the fuzzy relation takes a block-triangular form.



		1	1	1	5	5	4	2	6	3	3
		A	B	C	H	I	G	D	J	E	F
1	A	1	0,2	0,6	0	0	0,4	0,3	0,5	0,5	0,5
1	B	0,2	1	0,2	0	0	0,4	0,3	0,5	0,5	0,5
1	C	0,6	0,2	1	0	0	0,4	0,3	0,5	0,5	0,5
5	H	0	0	0	1	0,8	0	0	0	0,6	0,6
5	I	0	0	0	0,8	1	0	0	0	0,6	0,6
4	G	0	0	0	0	0	1	0	0,8	0	0
2	D	0	0	0	0	0	0	1	0	0,4	0,4
6	J	0	0	0	0	0	0	0	1	0	0
3	E	0	0	0	0	0	0	0	0	1	0,4
3	F	0	0	0	0	0	0	0	0	0,4	1

In this figure a class C_i is represented by its index $i = 1,2,\ldots,6$.

Fig. 28.7

†This property in prevented by certain authors under the name of the theorem of Szpilrajn. The introduction of the notion of the ordinal function of a graph avoids the rather delicate proof of this theorem. This is one of the advantages among many others of this important notion of ordinal function.

Figure 28,7 represents the results obtained in taking the total order

$$C_1 \succ C_5 \succ C_4 \succ C_2 \succ C_6 \succ C_3$$

with which one obtains a half-matrix of zeros below the diagonal of these blocks.

By choosing a total order in an ordinal function numbered from right to left, onewould obtain a half-matrix of zeros above the diagonal of the blocks.

This we summarize in an example generalizable to all cases conforming to the title of this subsection.

PROPERTIES OF THE PRINCIPAL FUZZY RELATIONS

	Reflexivity	*Anti reflexivity*	*Max-min transitivity*	*Min-max transitivity*	*Symmetry*	*Anti-symmetry*	*Does not possess circuits other than loops*

Preorder	*Yes*		*Yes*				
Similitude	*Yes*		*yes*		*Yes*		
Dissimilitude		*Yes*		*yes*	*Yes*		
Ressemblance	*Yes*				*Yes*		
Dissemblance		*Yes*			*Yes*		
Ordinate	*Yes*					*Yes*	*Yes*
Nonstrict order	*Yes*		*Yes*			*Yes*	*Yes*
Strict order		*Yes*	*Yes*			*yes*	*Yes*

29. COMMON MEMBERSHIP FUNCTIONS

In the tables that follow we have presented various continuous membership functions that are useful for representing numerical fuzzy subsets corresponding to the following fuzzy propositions:

$x$is small (29.1)-(29.7)

x is large (29.8)-(29,14)

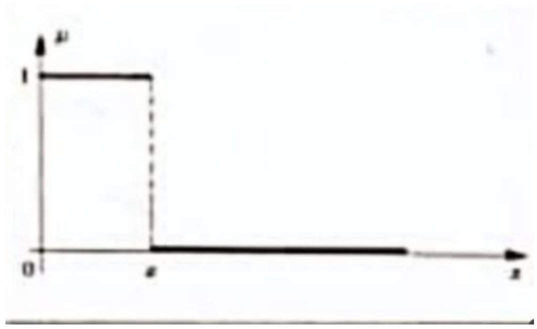
$|x|$is small (29.15)-(29.21)

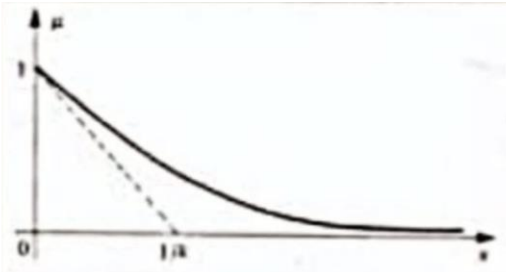
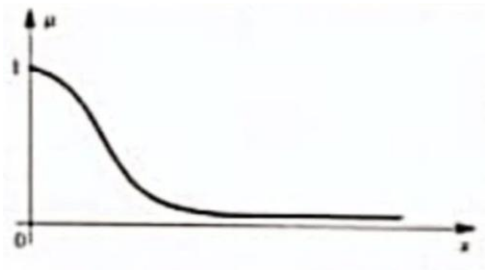
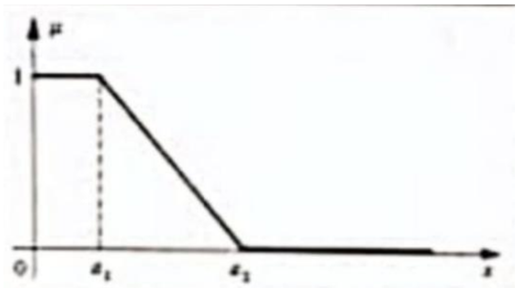
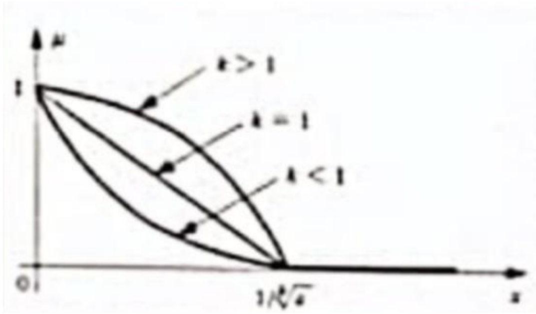
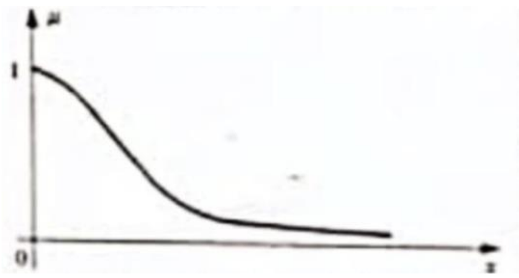
$|x|$is large (29.22)-(29.28)

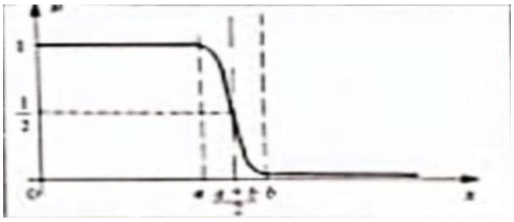
With respect to these one may construct numerical fuzzy subsets relative to two variables. We shall show how to proceed. Also in the same section we shall show how to analyze or synthesize transitive fuzzy relations.

REFERENCE SETS: $R^{+}.N$

MEMBERSHIP FUNCTION CORRESPONDING TO "x IS SMALL"

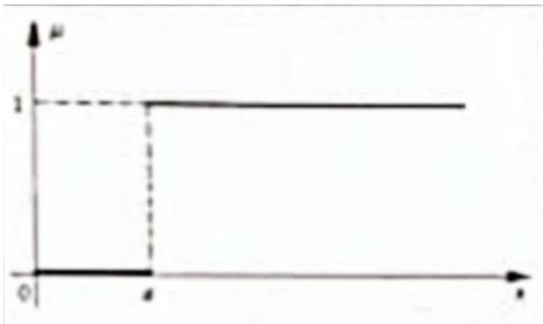
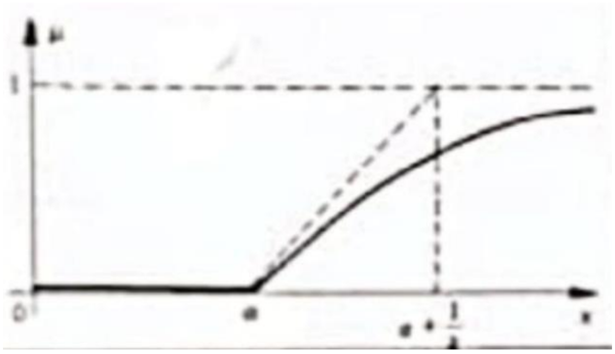
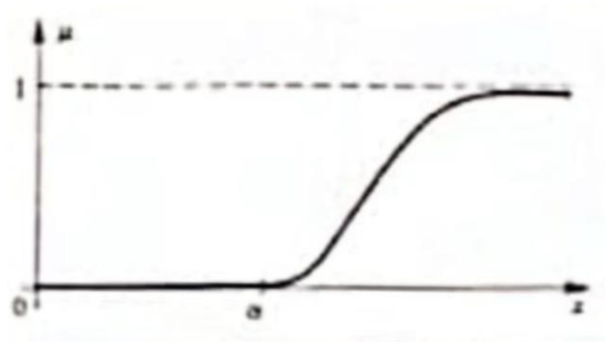
	Domain	Curve	Function
(29.1)	R^{+} N		$\mu(x) = 1, 0 \leq x \leq a$ $= 0, x > a$

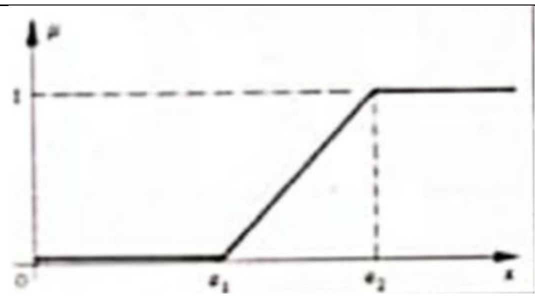
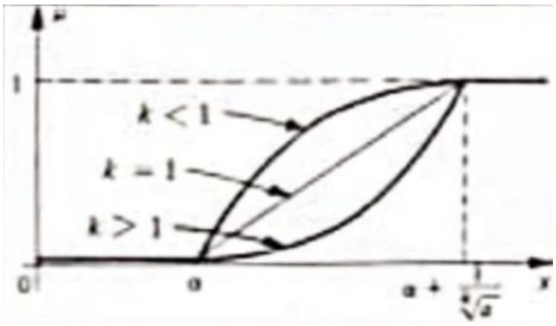
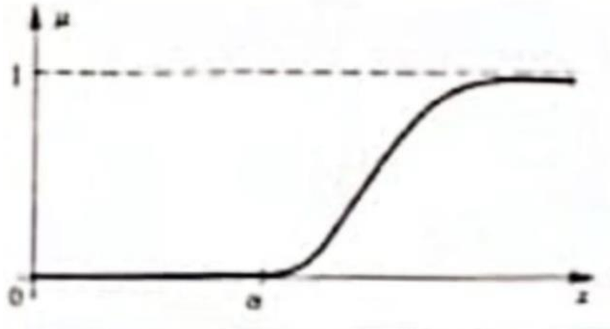
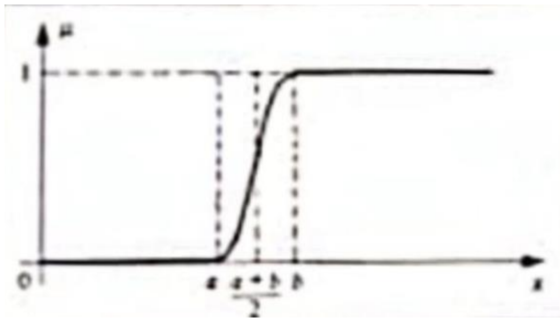
(29.2)	R^+ N		$\mu(x) = e^{-kx}, k > 0$
(29.3)	R^+ N		$\mu(x) = e^{-kx^2}, k > 0$
(29.4)	R^+ N		$\mu(x) = 1, 0 \le x \le a_1$ $= \dfrac{a_2 - x}{a_2 - a_1}, a_1 \le x \le a_2$ $= 0, a_2 \le x$
(29.5)	R^+ N		$\mu(x) = 1 - ax^2, 0 \le x \le \dfrac{1}{\sqrt[k]{a}}$ $= 0, \dfrac{1}{\sqrt[k]{a}} < x$
(29.6)	R^+ N		$\mu(x) = \dfrac{1}{1 + kx^2}, k > 1$

(29.7)	R^+ N		$\mu\,(x) = 1{,}0 \le x \le a$ $= \frac{1}{2} - \frac{1}{2}\sin\frac{x}{b-a}\left(x - \frac{a+b}{2}\right)$ $a \le x \le b$ $= 0, b < x$

REFERENCE SETS: R^+.N

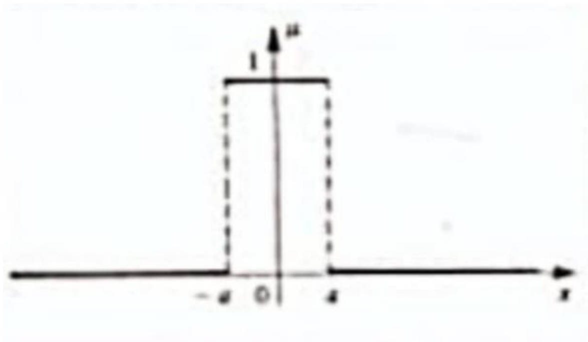
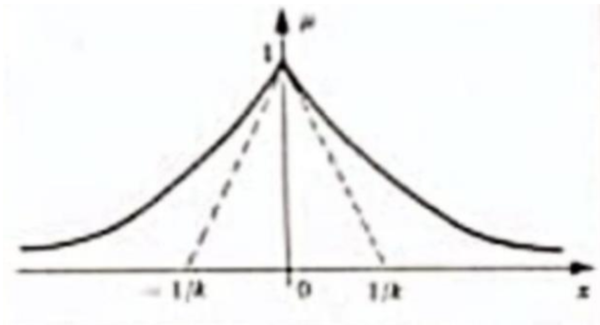
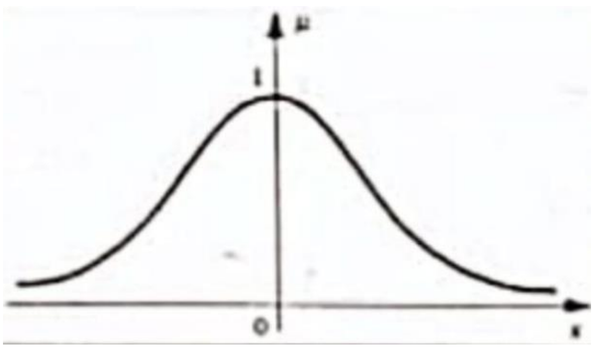
MEMBERSHIP FUNCTION CORRESPONDING TO “a IS LARGE”

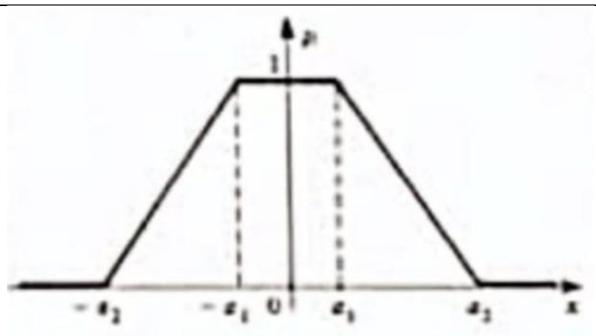
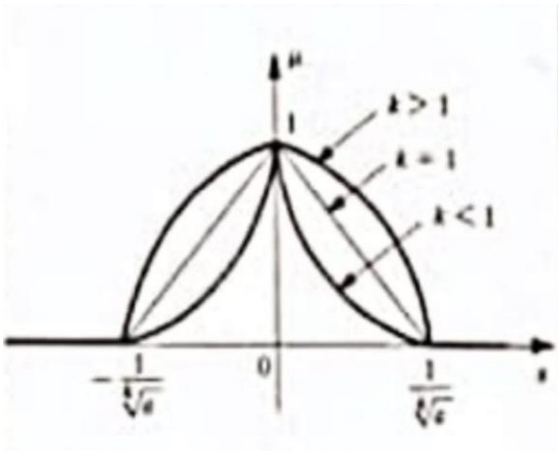
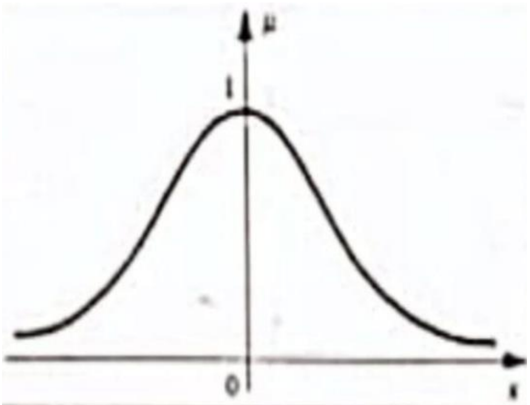
	Domain	Curve	Function
(29.8)	R^+ N		$\mu(x) = 0{,}0 \le x < a$ $= 1, a < x$
(29.9)	R^+ N		$\mu(x) = 0{,}0 \le x \le a$ $= 1 - e^{-k(x-a)}, a < x$ $x > 0$
(29.10)	R^+ N		$\mu(x) = 0{,}0 \le x \le a$ $= 1 - e^{-k(x-a)^2}, a \le x$ $k > 0$
(29.11)	R^+ N		$\mu(x) = 0{,}0 \le x \le a_1$ $= \frac{x-a_1}{a_2-a_1}, a_1 \le x \le a_2$

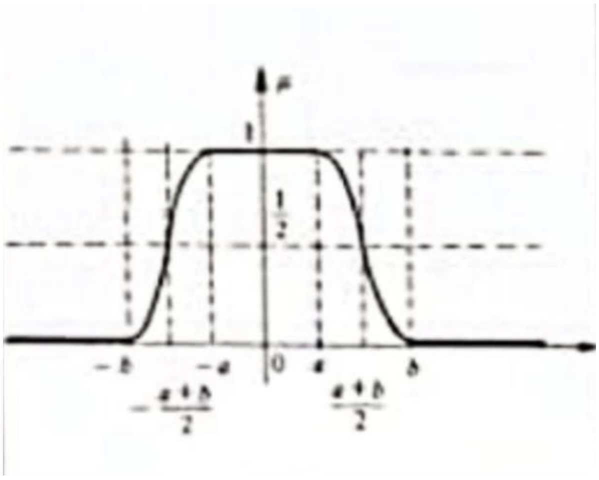
			$= 1, a_2 \le x$
(29.12)	R^+ N		$\mu(x) = 0, 0 \le x \le a$ $= a(x-a)^2, a \le x \le a + \frac{1}{\sqrt[k]{a}}$ $= 1, a + \frac{1}{\sqrt[k]{a}} \le x$
(29.13)	R^+ N		$\mu(x) = 0, 0 < x < a$ $= \frac{k(x-a)^2}{1+k(x-a)^2}, a \le x < \infty$
(29.14)	R^+ N		$\mu\ (x) = 0, 0 \le x \le a$ $= \frac{1}{2} - \frac{1}{2}\sin\frac{x}{b-a}\left(x - \frac{a+b}{2}\right)$ $a < x < b$ $= 01\ a < x$

REFERENCE SETS: R,Z

MEMBERSHIP FUNCTION CORRESPONDIND TO “$|X|$ $IS\ SMALL$”

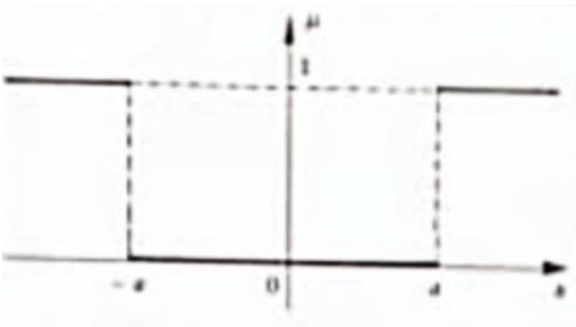
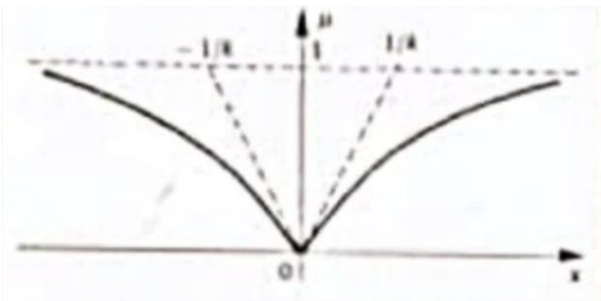
	Domain	Curve	Function
(29.15)	R Z		$\mu(x) = 0, -\infty \leq x < a$ $= 1, -a < x < a$ $= 0, a < x$
(29.16)	R Z		$\mu(x) = e^{kx}, -\infty < x < 0$ $= e^{-kx}, 0 < x < \infty$ $k > 1$
(29.17)	R Z		$\mu(x) = e^{-kx^2}$
(29.18)	R Z		$\mu(x) = 0, \infty \leq x \leq a_2$ $= \dfrac{a_2 + x}{a_2 - a_1}, -a_2 \leq x \leq a_1$ $= 1, a_2 \leq x < a_1$

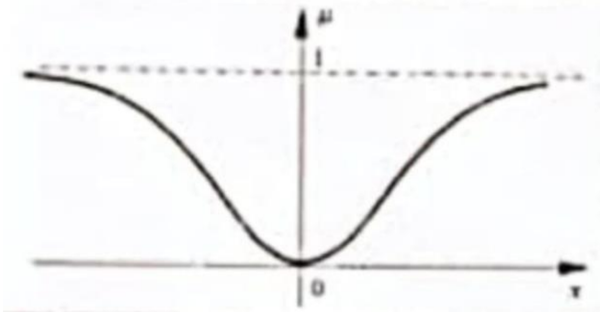
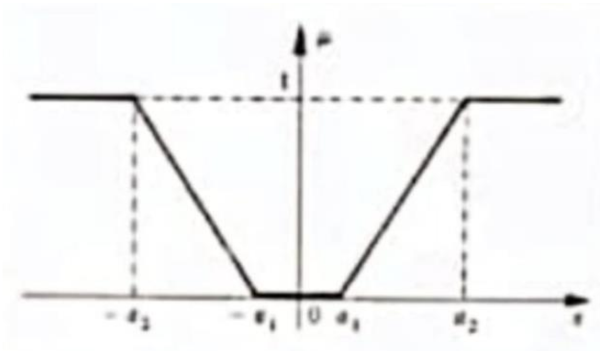
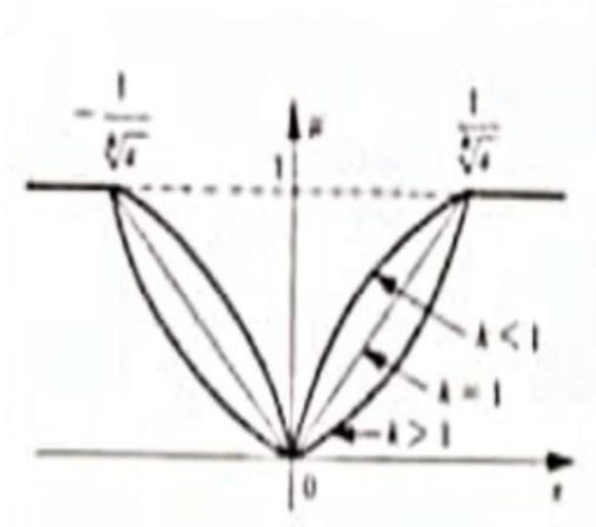
			$= \dfrac{a_2 - x}{a_2 - a_1}, a_1 < x < a_2$ $= 0, a_2 < x < \infty$
(29.19)	R Z		$\mu(x) = 0, -\infty < x \leq -\dfrac{1}{\sqrt[k]{a}}$ $= 1 - a(-x)^k, -\dfrac{1}{\sqrt[k]{a}} < x < 0$ $= 1 - a(x)^k, 0 < x < \dfrac{1}{\sqrt[k]{a}}$ $= 0, \dfrac{1}{\sqrt[k]{a}} < x < \infty$
(29.20)	R Z		$\mu(x) = \dfrac{1}{1 + kx^2}, k > 1$

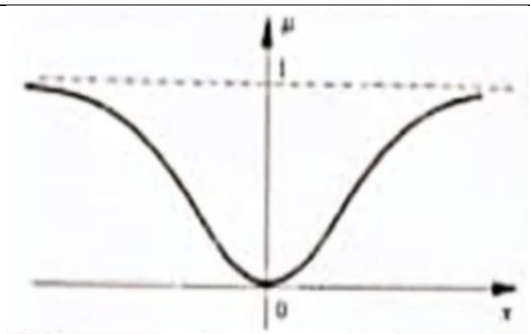
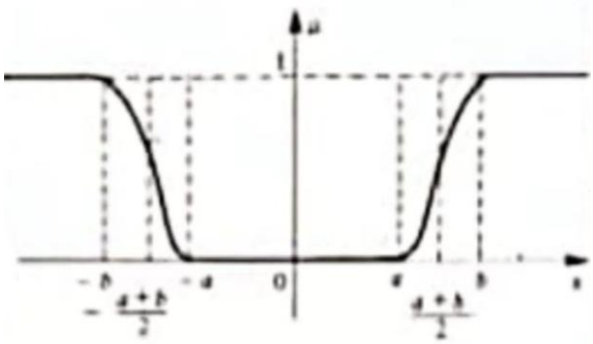
(29.21)	R Z		$\mu(x) = 0, -\infty < x < -b$ $= \frac{1}{2} + \frac{1}{2}\sin\frac{x}{b-a}\left(x + \frac{a+b}{2}\right), -b \le x < -a$ $= 1, -a < x < a$ $= \frac{1}{2} + \frac{1}{2}\sin\frac{x}{b-a}\left(x + \frac{a+b}{2}\right), -b \le x < -a$

REFERENCE SETS: R,Z

MEMBERSHIP FUNCTION TO "$|X|$ IS LARGE"

	Domain	Curve	Function
(29.22)	R Z		$\mu(x) = 1, -\infty \le x < -a$ $= 0, -a < x < a$ $= 1, a < x < \infty$
(29.23)	R Z		$\mu(x) = 1 - e^{kx}, -\infty < x < 0$ $= 1 - e^{-kx}, 0 < x < \infty$ $k > 1$
(29.24)	R Z		$\mu(x) = 1 - e^{-kx^2}$, k > 1

			
(29.25)	R Z		$\mu(x) = 1, -\infty \leq x \leq -a_2$ $= -\frac{x + a_1}{a_2 - a_1}, -a_2 \leq x \leq a_1$ $= 0, -a_1 \leq x < a_1$ $= \frac{x - a_1}{a_2 - a_1}, a_1 < x < a_2$ $= 1, a_1 < x < \infty$
(29.26)	R Z		$\mu(x) = 1, -\infty < x \leq -\frac{1}{\sqrt[k]{a}}$ $= a(-x)^k, -\frac{1}{\sqrt[k]{a}} < x < 0$ $= a(x)^k, 0 < x < \frac{1}{\sqrt[k]{a}}$ $= 0, \frac{1}{\sqrt[k]{a}} < x < \infty$
(29.27)	R Z		$\mu(x) = \frac{kx^2}{1 + kx^2}$ $= \frac{1}{1 + \frac{1}{kx^2}}, k > 1$

			
(29.28)	R Z		$\mu(x) = 1, -\infty < x < -b$ $= \frac{1}{2} - \frac{1}{2} \sin \frac{x}{b-a}\left(x + \frac{a+b}{2}\right), -b \leq x < -a$ $= 0, -a < x < a$ $= \frac{1}{2} + \frac{1}{2} \sin \frac{x}{b-a}\left(x + \frac{a+b}{2}\right), a \leq x < b$ $= 1, b < x < \infty$

REFERENCE SETS $R^+ \times R^+, R \times R, N \times N, Z \times Z$

A. Cylindrical membership functions,† of the type

(29.29) $$\mu(x, y) = f(x^2 + y^2)$$

corresponding to "$x^2 + y^2$ has property$\mathcal{P}$.

Take the curves and functions (29.1)-(29.14) and replace

$$x\ by\ \rho = \sqrt{x^2 + y^2}$$

For (29.1)-(29.14), property $\mathcal{P}$will be

$$x^2 + y^2\ is\ small$$
$$or\ x\ and\ y\ are\ small$$

For (28.8)-(29.14), property $\mathcal{P}$will be

$$x^2 + y^2 \; is \; large$$
$$or \; x \; and \; y \; are \; large$$

Example. With reference to (29.6), one may see

(29.30) $$\mu(x, y) = \frac{1}{1 + k(x^2 + y^2)}$$

B. Hyperbolic membership functions, of the type

(29.31) $$\mu(x, y) = f(|y - x|)$$

Or

(29.32) $$\mu(x, y) = f(|y^2 - x^2|)$$

corresponding to $|y - x|$ or $|y^2 - x^2|$ have property$\mathcal{P}$.

Take the curves and functions of (29.1)-(29,14) and replace

(29.33) $$x \; by \; \rho = |y - x|$$

or

$$\rho = \sqrt{|y^2 - x^2|}$$

For (29.1)-(29.7), property$\mathcal{P}$ will be

$$y \; is \; very \; near \; x$$

For (29.8)-(29.14), it will be

$$y \; is \; very \; different \; than \; x$$

One may also me $\rho = |y - kx|$ with & sufficiently large and reversing the opposite properties above.

Remark.- One knows that

(29.34) $$e^{-u} = \frac{1}{1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots}$$

Thus, the functions.

(29.35) $$e^{-u}\ et\frac{1}{1+u}$$

†We shall find it convenient to describe it so,

will give similar results, taking these as membership functions when $u=\phi(x), u=\left(\sqrt{x^2}+y^2\right),\ u=\phi(\sqrt{|y^2-x^2|})$ and similarly for the other variables.

Determination of the property of max-min transitivity in the case of continuous membership function of a relation. It is generally very easy to evaluate a membership function $\mu_{\underline{\mathcal{R}}}(x,y)$ if it presents one of the following properties:

Reflexivity,

Symmetry,

Antisymmetry.

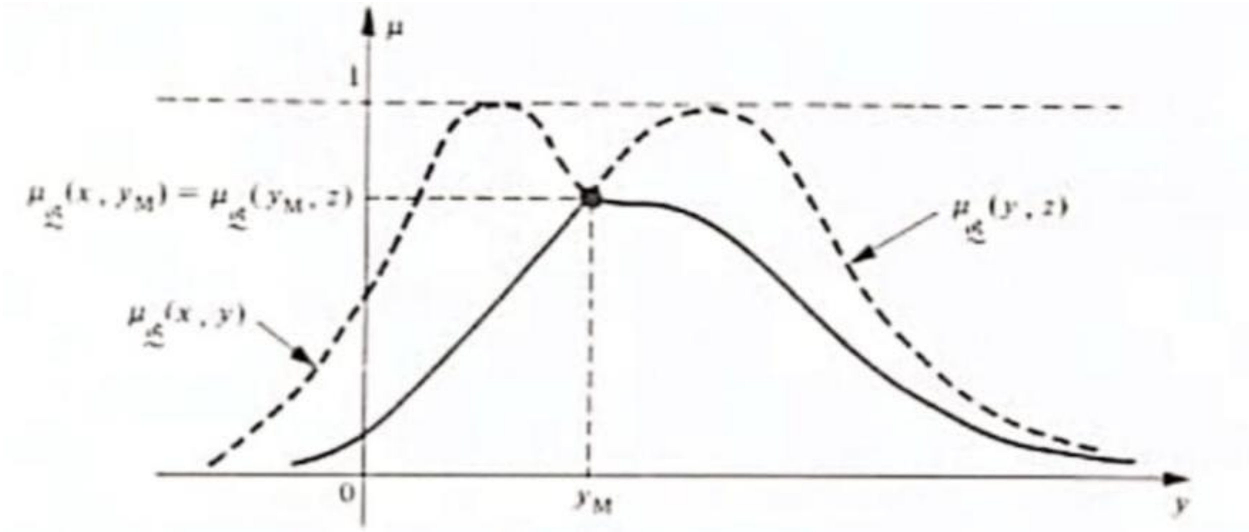
But it is generally much less easy to be concerned with transitivity. We shall first consider max-min transitivity, then max-product transitivity.†

Recall that max-min transitivity is expressed by the property

(29.36) $$\mu_{\underline{\mathcal{R}}}(x,z)>V_y\left|\mu_{\underline{\mathcal{R}}}(x,y)\wedge\mu_{\underline{\mathcal{R}}}(y,z)\right|.$$

.

In Figure 29.1 we have shown how to obtain the member on the right of (29.36), In this example there is a single intersection point between $\mu_{\underline{\mathcal{R}}}(x,y)$ and$\mu_{\underline{\mathcal{R}}}(y,z)$ when x and z are taken as parameters, there may exist several in other cases, but each time one determines the maximum y_m. In the sequel it is convenient to proceed in the following fashion:



The solid curve gives

$\mu_{\underline{\mathcal{R}}}(x,y) \wedge \mu_{\underline{\mathcal{R}}}(y,z)$

Fig 29.1

(1) Determine the point y_M as a function of x and z such that

(29.37) $$\mu_{\underline{\mathcal{R}}}(x,y_M) = \mu_{\underline{\mathcal{R}}}(y_M,z)$$

(2) Substitute the value of y_M as a function of x and z into $\mu_{\underline{\mathcal{R}}}(x,y_M)$ or in $\mu_{\underline{\mathcal{R}}}(y_M,z)$, this gives a function $\lambda(x,z)$.

(3) Compare $\lambda(x,z)$ to $\mu_{\underline{\mathcal{R}}}(x,z)$. If

(29.38) $$\forall(x,z) : \ \mu_{\underline{\mathcal{R}}}(x,z) > \lambda(x,z)$$

then the relation $\underline{\mathcal{R}}$ is transitive. If

†One may easily pass to proofs involving min-max tramitivity and min-sum transitinity, or even ordinary min addition,

(29.39) $$\exists\ (x,z) : \ \mu_{\underline{\mathcal{R}}}(x,z) < \lambda(x,z)$$

then the relation $\underline{\mathcal{R}}$ is not transitive,

We consider several examples.

Example 1. Consider the fuzzy relation $\underline{\mathcal{R}}$ defined for $x \in R^+$ and $y \in R^+$:

(29.40)
$$\begin{aligned} \mu_{\underline{\mathcal{R}}}(x,y) &= e^{-x}, y < x \\ &= 1, y = x \\ &= e^{-y}, y > x \end{aligned}$$

In Figure 29.2 we have represented, as a function of y, the function $\mu_{\underline{\mathcal{R}}}(x,y)$(x taken as a parameter) and $\mu_{\underline{\mathcal{R}}}(x,y)$(z taken as a parameter) in the cases where $x < z$ (Figure 29.2a) and $x > z$(Figure 29.2b). In these figures $ABCD$ represents $\mu_{\underline{\mathcal{R}}}(x,y)$ (with xas parameter) and$A'B'CD$ represents $\mu_{\underline{\mathcal{R}}}(y,z)$ (with z as parameter).

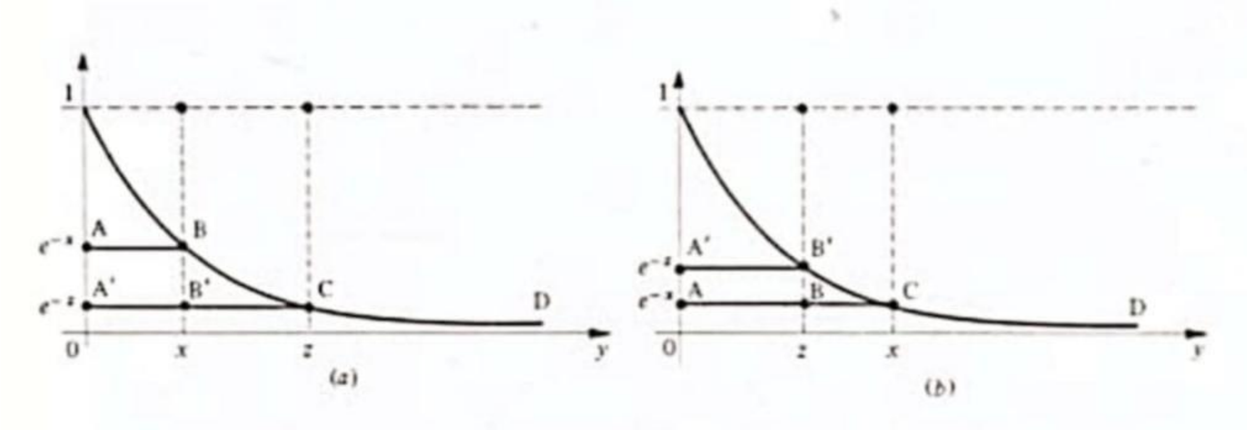


Fig. 29.2

In the $x < z$ case the max-min is equal to e^{-z}, and in the $x > z$ case the max-min is equal toe^{-x}. Thus, one may write

(29.41)
$$\begin{aligned}\lambda(x,z) &= e^{-z}, x \leq z\\ &= e^{-x}, x \geq z\end{aligned}$$

Compare $\lambda(x,z)$ with $\mu_{\underline{\mathcal{R}}}(x,z)$ given by (29.40) where $z$replaces y:

(29.42)
$$\begin{aligned}\mu_{\underline{\mathcal{R}}}(x,z) &= e^{-z}, x < z\\ &= 1\ , x = z\\ &= e^{-x}, x > z\end{aligned}$$

By comparing one sees that

(29.43)
$$\mu_{\underline{\mathcal{R}}}(x,z) = \lambda(x,z), x \neq z,$$

(29.44)
$$\mu_{\underline{\mathcal{R}}}(x,z) > \lambda(x,z), x = z,$$

Then$\underline{\mathcal{R}}$ is a transitive relation. We note that this relation is a similitude relation.

In Figure 29.3 we have represented with the aid of a matrix the fuzzy relation corresponding to (29.40) but using N instead of R^+. This shows the particular arrangement of the values of the membership function. The reader should involve himself in applying and verifying transitivity in this case using (29.36). The row-by-column max- min operation will permit one to check (29.43) and (29.44),

$x \backslash y$	0	1	2	3	4	5	6	7	...
0	1	e^{-1}	e^{-2}	e^{-3}	e^{-4}	e^{-5}	e^{-6}	e^{-7}	...
1	e^{-1}	1	e^{-2}	e^{-3}	e^{-4}	e^{-5}	e^{-6}	e^{-7}	...
2	e^{-2}	e^{-2}	1	e^{-3}	e^{-4}	e^{-5}	e^{-6}	e^{-7}	...
3	e^{-3}	e^{-3}	e^{-3}	1	e^{-4}	e^{-5}	e^{-6}	e^{-7}	...
4	e^{-4}	e^{-4}	e^{-4}	e^{-4}	1	e^{-5}	e^{-6}	e^{-7}	...
5	e^{-5}	e^{-5}	e^{-5}	e^{-5}	e^{-5}	1	e^{-6}	e^{-7}	...
6	e^{-6}	e^{-6}	e^{-6}	e^{-6}	e^{-6}	e^{-6}	1	e^{-7}	...
7	e^{-7}	e^{-7}	e^{-7}	e^{-7}	e^{-7}	e^{-7}	e^{-7}	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 29.3

Example 2. Consider the fuzzy relation $\underset{\sim}{\mathcal{R}}$defined for$x \in R$and $y \in R$:

(29.45) $$\mu_{\underset{\sim}{\mathcal{R}}}(x,y) = e^{-(x-y)^2} :$$

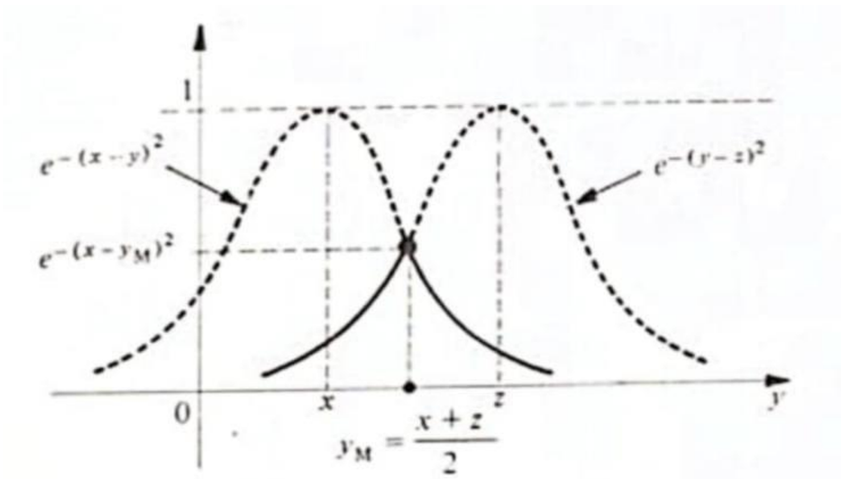


Fig. 29.4

One finds easily

(29.46) $$e^{-(x-y_M)^2} = e^{-(y_M-z)^2}$$

Thus

(29.47) $$x - y_M = y_M - z$$

or

(29.48) $$y_M = \frac{x+z}{2}$$

See Figure 29.4. Putting this value in the member on the right of (29.45), we have

(29.49) $$\begin{aligned}\lambda(x,z) &= e^{-(x-y_M)^2}\\ &= e^{-\frac{(x-z)^2}{4}}\end{aligned}$$

And we see that

(29.50) $$\forall(x,z)\colon e^{-(x-z)^2} \le e^{-\frac{(x-z)^2}{4}}$$

that is

(29.51) $$\forall(x,z):\ \mu_{\underline{\mathcal{R}}}(x,z) \le \lambda(x,z)$$

Thus, this relation$\underline{\mathcal{R}}$ is not transitive. We note that sometimes this is a resemblancerelation.

If Figure 29.5 we have represented the corresponding relation $\underline{\mathcal{R}}$but using N instead of R^+.

$x \backslash y$	0	1	2	3	4	5	6	7	...
0	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}	e^{-25}	e^{-36}	e^{-49}	...
1	e^{-1}	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}	e^{-25}	e^{-36}	...
2	e^{-4}	e^{-1}	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}	e^{-25}	...
3	e^{-9}	e^{-4}	e^{-1}	1	e^{-1}	e^{-4}	e^{-9}	e^{-16}	...
4	e^{-16}	e^{-9}	e^{-4}	e^{-1}	1	e^{-1}	e^{-4}	e^{-9}	...
5	e^{-25}	e^{-16}	e^{-9}	e^{-4}	e^{-1}	1	e^{-1}	e^{-4}	...
6	e^{-36}	e^{-25}	e^{-16}	e^{-9}	e^{-4}	e^{-1}	1	e^{-1}	...
7	e^{-49}	e^{-36}	e^{-25}	e^{-16}	e^{-9}	e^{-4}	e^{-1}	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 29.5

Example 3. Consider the fuzzy relation $\underline{\mathcal{R}}$ defined for x\in R^{+} y\in R^{+}

(29.52) $$\begin{aligned}\mu_{\underline{\mathcal{R}}}(x,y) &= \frac{xy}{1+xy}, y > x\\ &= 0, y \le x\end{aligned}$$

Figure 29.6 shows that the min-max corresponds to $y_M = z$. Whence

(29.53) $$\begin{aligned}\lambda(x,y_M) &= \lambda(x,z),\\ &= \frac{xz}{1+xz}, z > x\\ &= 0, z < x\end{aligned}$$

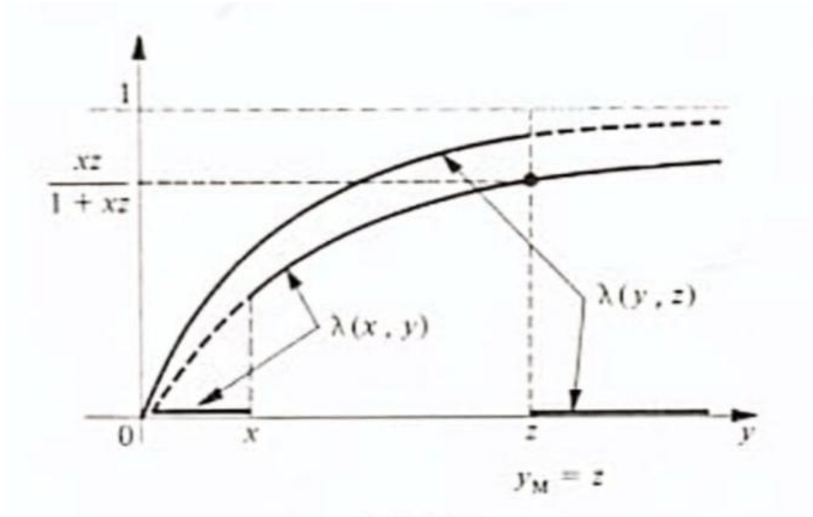


Fig. 29.6

We see that

(29.54) $$\mu_{\underline{\mathcal{R}}}(x,z) = \lambda(x,z)$$

Thus the relation$\underline{\mathcal{R}}$ is indeed transitive. One may also verify that this telation is a total fuzzy order.

Figure 29.7 represents the corresponding relation with R^+ replaced by N.

Remark on the synthesis of a transitive fuzzy relation. If the analysis of a fuzzy relation in R or R^+ is not very easy, as we have seen, its synthesis is even more difficult except in ceftain very simple particular cases. Thus, a good method consists in carrying out the synthesis of the relation in N, and then passing from there to$R$ or R^+.

The decomposition theorem for a similitude relation (Section 27) and that for a perfect order relation (Section 28) allow the easy synthesis of the corresponding relations. One may provide an algorithm.

Algorithm for construction of a fuzzy transitive relation in a denumerable set.

(1) We are given a sequence, finite or not, of numbers $x_i \in [0,1]$, strictly ordered on i:

(29.55) $$1 > a_1 > a_2 > \cdots > a_r > \cdots > 0.$$

$x \backslash y$	0	1	2	3	4	5	6	7	...
0	0	0	0	0	0	0	0	0	...
1	0	0	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	...
2	0	0	0	$\frac{6}{7}$	$\frac{8}{9}$	$\frac{10}{11}$	$\frac{12}{13}$	$\frac{14}{15}$	...
3	0	0	0	0	$\frac{12}{13}$	$\frac{15}{16}$	$\frac{18}{19}$	$\frac{21}{22}$	...
4	0	0	0	0	0	$\frac{20}{21}$	$\frac{24}{25}$	$\frac{28}{29}$	...
5	0	0	0	0	0	0	$\frac{30}{31}$	$\frac{35}{36}$	...
6	0	0	0	0	0	0	0	$\frac{42}{43}$	...
7	0	0	0	0	0	0	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 29.7

Step-by-step construct a transitive ordinary graph by enriching the arcs, always maintaining transitivity. To each passage from a transitive graph to a transitive graph richer in arcs, this constituting a step, associate with the corresponding arcs of the fuzzy graph the value a_{i+1}that follows the preceding a_i, value. The finite fuzzy graph obtained by stopping at step i is transitive; if the procedure is not stopped at a finite number, one obtains an infinite graph that is transitive.

Example. Consider the infinite sequence:

(29.56) $$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \cdots > \frac{1}{r} > \cdots > 0.$$

We propose to construct a transitive and antireflexivefuzzy graph having perfect antisymmetry. The construction will be carried out according to the order indicated in Figure 29.8, where the addition of arcs is arbitrary except that, at each step, transitivity must be maintained, In Figure 29.9 one may see how to construct the fuzzy graph.

The same procedure may be used for preorders, similitudes, orders, etc.

From the matrix obtained one may see the corresponding relation$\underline{\mathcal{R}}$ of to be obtained for R^+ and eventually for R. This is not, evidently, always easy.

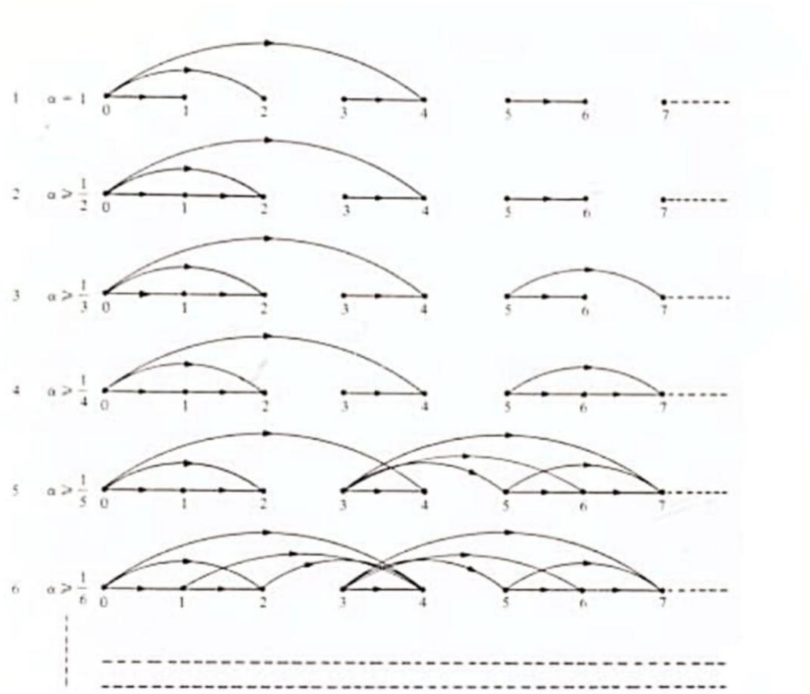


Fig. 29.8

$x \backslash y$	0	1	2	3	4	5	6	7	...
0	0	1 1	1 1		1 1				...
1	0	0	$\frac{1}{2}$ 2		$\frac{1}{6}$ 6				...
2	0	0	0		$\frac{1}{6}$ 6				...
3	0	0	0	0	1 1	$\frac{1}{5}$ 5	$\frac{1}{5}$ 5	$\frac{1}{5}$ 5	...
4	0	0	0	0	0				...
5	0	0	0	0	0	0	1 1	$\frac{1}{3}$ 3	...
6	0	0	0	0	0	0	0	$\frac{1}{4}$ 4	...
7	0	0	0	0	0	0	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 29.9

UNIT IV

FUZZY LOGIC

29. INTRODUCTION

To associate the word fuzzy with the word logic is shocking. Logic, in the ordinary sense of the word, is a conceptualization of the mechanisms of thought, one that may never be fuzzy, but always rigorous and formal. Mathematicians researching these mechanisms of thought have noted, however, that it is not a matter of having, in fact, one unique logic (for example, Boolean logic). This unit stimulate the imagination of the readers so that they will go much further than what is modestly presented here.

30. CHARACTERISTIC FUNCTION OF A FUZZY SUBSET FUZZY VARIABLES

Let $\mu_{\underset{\sim}{A}}\ (x)$ be the membership function of the element x in the fuzzy subset $\underset{\sim}{A}$. In Section 2-8 we have defined the principal operations that may be realized in considering fuzzy subsets with the same reference set.

In the present chapter, we shall suppose that the membership set is always (30.1) M = [0,1].

Previously, we have recalled how to carry out the operations of a binary Boolean algebra with respect to the algebraic operations of ordinary subsets.

We shall use the following notation:

(30.2) $$a = \mu_A\ (\mathrm{x}),\ b = \mu_B\ (\mathrm{x}), \text{etc.}$$

We know that in Boolean binary algebras the variables, such as a, b, ... may take only the values 0 or 1. The correspondence between the operations of ser theory and those of Boolean binary algebra is reviewed below:

	Subsets	Corresponding operations
(30.3)	$A \cap B.$	a, b.
(30.4)	$A \cup B.$	a + b.
(30.5)	$\tilde{A} = C_E\mathrm{A}\,.$	$\tilde{a}.$
(30.6)	$A + B = (\tilde{A} \cap B) \cup (A \cap \tilde{B}$	$a + b = \tilde{a}\,, b + a, \tilde{b}$

The principal correspondences that we see in (32.3) – (32.6) constitute a didactic introduction to what follows, and will hold not only for boolean characteristic functions and membership functions with M = [0,1] but also for fuzzy functions with M = [0,1].

Let *x* be an element of the reference set E and let $\underset{\sim}{A}$, $\underset{\sim}{B}$, . . . be fuzzy subsets of this reference set. Put

(30.7) $\underset{\sim}{a} = \mu_{\underset{\sim}{A}}(x),\ \underset{\sim}{b} = \mu_{\underset{\sim}{B}}(x)$ ..... $\underset{\sim}{a}\,, \underset{\sim}{b}$ . . . . ∈M = [0,1].

We shall define, with respect to what was given in Sections 2-8, the following operations for the quantities, a, b . . . . .

(30.8) $$\underset{\sim}{a} \wedge \underset{\sim}{b} = \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}\right).$$

(30.9) $$\underset{\sim}{a} \vee \underset{\sim}{b} = \text{MAX}\left(\underset{\sim}{a}, \underset{\sim}{b}\right).$$

(30.10) $$\bar{\underset{\sim}{a}} = 1 - \underset{\sim}{a}\,.$$

31. *CHARACTERISTIC FUNCTION OF FUZZY SUBSET*

(31.11) $$\underset{\sim}{a} \oplus \underset{\sim}{b} = \left(\bar{\underset{\sim}{a}} \wedge \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \wedge \bar{\underset{\sim}{b}}\right).$$

(31. 12) $$\left.\begin{array}{l} \underset{\sim}{a} \wedge \underset{\sim}{b} = \underset{\sim}{b} \wedge \underset{\sim}{a}\,. \\ \underset{\sim}{a} \vee \underset{\sim}{b} - \underset{\sim}{b} \vee \underset{\sim}{a}\,. \end{array}\right\} \text{ commutativity}$$

(31.13)

(31.14) $$\left.\begin{array}{l} \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) \wedge \underset{\sim}{c} = \underset{\sim}{a} \wedge \left(\underset{\sim}{b} \wedge \underset{\sim}{c}\right). \\ \left(\underset{\sim}{a} \vee \underset{\sim}{b}\right) \vee \underset{\sim}{c} = \underset{\sim}{a} \vee \left(\underset{\sim}{b} \vee \underset{\sim}{c}\right). \end{array}\right\} \text{ associativity}$$

(31.15)

(31.16) $$\left.\begin{array}{l} \underset{\sim}{a} \wedge \underset{\sim}{a} = \underset{\sim}{a}\,. \\ \underset{\sim}{a} \vee \underset{\sim}{a} = \underset{\sim}{a}\,. \end{array}\right\} \text{ idempotence}$$

(31.17)

(31.18) $$\left.\begin{array}{l} \underset{\sim}{a} \wedge \left(\underset{\sim}{b} \vee \underset{\sim}{c}\right) = \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{c}\right). \\ \underset{\sim}{a} \vee \left(\underset{\sim}{b} \wedge \underset{\sim}{c}\right) = \left(\underset{\sim}{a} \vee \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \vee \underset{\sim}{c}\right). \end{array}\right\} \text{ Distributivity}$$

(31.19)

(31.20) $$\underset{\sim}{a} \wedge 0 = 0.$$

(31.21) $$\underset{\sim}{a} \vee 0 = \underset{\sim}{a}.$$

(31.22) $$\underset{\sim}{a} \wedge 1 = \underset{\sim}{a}.$$

(31.23) $$\underset{\sim}{a} \vee 1 = 1.$$

(31.24) $$\left(\bar{\underset{\sim}{a}}\right) = \underset{\sim}{a}.$$

(31.25) $$\left.\begin{array}{l} \overline{\underset{\sim}{a} \wedge \underset{\sim}{b}} = \bar{\underset{\sim}{a}} \vee \bar{\underset{\sim}{b}}\,. \\ \overline{\underset{\sim}{a} \vee \underset{\sim}{b}} = \bar{\underset{\sim}{a}} \wedge \bar{\underset{\sim}{b}}\,. \end{array}\right\} \begin{array}{c} \text{Do Morgan's theorems} \\ \text{generalized to the case where } M = [0,1]. \end{array}$$

(31.26)

The proofs of all these formulas are trivial except perhaps those of (31.18), (31.19), (31.25), and 31.26).

We shall prove (31.18). For this, we suppose that the quantities $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ have their values in the following three distinct total orders (it is not useful to consider six):

(31.27) **1)** $0 \le \underset{\sim}{a} \le \underset{\sim}{b} \le \underset{\sim}{c} \le 1$. **2)** $0 \le \underset{\sim}{b} \le \underset{\sim}{c} \le \underset{\sim}{a} \le 1$ and **3)** $0 \le \underset{\sim}{c} \le \underset{\sim}{a} \le \underset{\sim}{b} \le 1$.

We have

(31.28) $$1)\ \underset{\sim}{a} \wedge \left(\underset{\sim}{b} \vee \underset{\sim}{c}\right) = \text{MIN}\left[\underset{\sim}{a}, \text{MAX}\left(\underset{\sim}{b}, \underset{\sim}{c}\right)\right]$$

$$= \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{c}\right) = \underset{\sim}{a}.$$

(31.29) $$\left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{c}\right) = \text{MAX}\left[\text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}\right) \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{c}\right)\right]$$

$$= \text{MAX}\left(\underset{\sim}{a}, \underset{\sim}{a}\right) = \underset{\sim}{a}.$$

(31.30) $$2)\ )\ \underset{\sim}{a} \wedge \left(\underset{\sim}{b} \vee \underset{\sim}{c}\right) = \text{MIN}\left[\underset{\sim}{a}, \text{MAX}\left(\underset{\sim}{b}, \underset{\sim}{c}\right)\right]$$

$$= \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{c}\right) = \underset{\sim}{c}.$$

$$\left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{c}\right) = \text{MAX}\left[\text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}\right), \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{c}\right)\right]$$

$$= \text{MAX}\left(\underset{\sim}{b}, \underset{\sim}{c}\right) = \underset{\sim}{c}.$$

$$3)\ \underset{\sim}{a} \wedge \left(\underset{\sim}{b} \vee \underset{\sim}{c}\right) = \text{MIN}\left[\underset{\sim}{a}, \text{MAX}\left(\underset{\sim}{b}, \underset{\sim}{c}\right)\right]$$

$$= \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}\right) = \underset{\sim}{a}.$$

$$\left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{c}\right) = \text{MAX}\left[\text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}\right) \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{c}\right)\right]$$

$$= \text{MAX}\left(\underset{\sim}{a}, \underset{\sim}{c}\right) = \underset{\sim}{a}.$$

We shall prove $\overline{\underset{\sim}{a} \wedge \underset{\sim}{b}} = \overline{\underset{\sim}{a}} \vee \overline{\underset{\sim}{b}}$.

Let $0 \le \underset{\sim}{a} < \underset{\sim}{b} \le \underset{\sim}{c} \le 1$

$$\text{MAX}\left[\left(1 - \underset{\sim}{a}\right), \left(1 - \underset{\sim}{b}\right)\right] = 1 - \underset{\sim}{a}$$

$$= \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}\right) = \underset{\sim}{a}.$$

$$\text{MAX}\left[\left(1 - \underset{\sim}{a}\right), \left(1 - \underset{\sim}{b}\right)\right] + \text{MIN}\left[\left(\underset{\sim}{a}, \underset{\sim}{b}\right)\right] = 1 - \underset{\sim}{a} + \underset{\sim}{a} = 1 \text{ then}$$

$$\text{MAX}\left[\left(1-\underset{\sim}{a}\right),\left(1-\underset{\sim}{b}\right)\right]=\ 1\text{ - MIN}\left[\underset{\sim}{a},\underset{\sim}{b}\right].$$

$$\overline{\underset{\sim}{a}}\text{ v }\overline{\underset{\sim}{b}}=\overline{\underset{\sim}{a}\wedge\underset{\sim}{b}}\,.$$

Very important remark. Properties (32.12) – 32.26) are all the properties of a Boolean binary algebra with the following two exceptions.

$$a\,.\,\overline{a}=0 \text{ and } a\ \dot{+}\ \overline{a}=1$$

for which the corresponding expressions are not satisfied:

$\underset{\sim}{a}\ \wedge\overline{\underset{\sim}{a}}\neq 0$, except for $\underset{\sim}{a}=0$ or $\underset{\sim}{a}=1$.

$\underset{\sim}{a}\text{ v }\overline{\underset{\sim}{a}}\neq 1$, except for $\underset{\sim}{a}=0$ or $\underset{\sim}{a}=1$.

Because of this, the structure obtained over the variables $\underset{\sim}{a}\,,\underset{\sim}{b}$ …. for the operations ∧, v and ⁻ may not be considered as constituting an algebra in the sense given to this word in modern mathematics. Let it be understood also that the world *algebra*, as many other words in the language of mathematics, is not employed by all in the same sense.

Fuzzy variables. Functions of fuzzy variables. The variables $\varrho,\mathcal{R},\ \ldots\in[0,1]$ will be called fuzzy variables + in the present theory. The functions constructed with the aid of these variables will be called functions of fuzzy variables under the following condition.

Let $\underset{\sim}{f}\ \left(\underset{\sim}{a},\underset{\sim}{b},\ldots.\right)$ be a function $\varrho,\mathcal{R},$ …. In order that this function may be called a function of fuzzy it is necessary and sufficient that $\underset{\sim}{f}$ depend only on fuzzy variables and that

Theorem I. *If* $(\varrho,\mathcal{R},\ldots)$ contains only fuzzy variables and the operators v, ∧, and, $-$, then ***0*** < $\underset{\sim}{f}$ < *1*is always satisfied.

Proof. This is evident. Each of the operation ∧, v, $-$ on the variables $\varrho,\mathcal{R},\ \ldots\in[0,1]$ cannot produce a result outside the limits 0 and 1. Submitting such a result to the operations ∧, v, $-$ with other results of the same nature cannot produce a result outside these limits.

Simplification of functions of fuzzy variables. Functions of fuzzy variables may not be, as are Boolean binary functions, the objects of truth tables permitting an ordered analysis (In the order of binary numbers) of these functions. Nor may they be simplified easily, as are Boolean functions, because of the absence of the two properties (32.39) and (42.40). Also because of this, one may not decompose these functions in a disjunctive canonical form (with the aid of the minterms) or in a conjunctive canonical form (with the aid of maxterms.

We shall see several examples of simplifications:

$$\underset{\sim}{f}\left(\underset{\sim}{a},\ \underset{\sim}{b},\right) = \underset{\sim}{a} \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right)$$

$$= \underset{\sim}{a} \wedge \left(1 \vee \underset{\sim}{b}\right). \text{ after (31.18) and (31.22) since}$$

$$\underset{\sim}{a} \wedge \left(1 \vee \underset{\sim}{b}\right) = \left(\underset{\sim}{a} \wedge 1\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right)$$

$$= \underset{\sim}{a} \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right)$$

$$= \underset{\sim}{a} \wedge 1. \qquad \text{after (31.23)}$$

$$= \underset{\sim}{a}. \qquad \text{after (31.22).}$$

Also

$\underset{\sim}{a} \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) = \underset{\sim}{a}$. This is called the property of absorption.

In a similar fashion one may show that

$\underset{\sim}{a} \wedge \left(\underset{\sim}{a} \vee \underset{\sim}{b}\right) = \underset{\sim}{a}$. This is the dual form of the property of absorption.

We consider another example:

$$\underset{\sim}{f}\left(\underset{\sim}{a},\ \underset{\sim}{b},\ \underset{\sim}{c}\right) = \left(\underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}}\right) \vee \left(\overline{\underset{\sim}{a}} \wedge (\underset{\sim}{b} \vee \underset{\sim}{c}\right) \vee \overline{\underset{\sim}{a}} \vee \left((\underset{\sim}{b} \wedge \underset{\sim}{c}\right)$$

$$= \underbrace{\left(\underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}}\right)}_{(1)} \vee \underbrace{\left(\overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}}\right)}_{(2)} \vee \underbrace{\left(\underset{\sim}{a} \wedge \underset{\sim}{c}\right)}_{(3)} \vee \underbrace{\overline{\underset{\sim}{a}}}_{(4)} \vee \underbrace{\left((\underset{\sim}{b} \wedge \overline{\underset{\sim}{c}}\right)}_{(5)}$$

$$= (b \wedge \overline{c}) \vee \overline{a}$$

It is unnecessary to recall the important role of parentheses.

FUZZY LOGIC

We know that the number of distinct boolean functions obtained with the aid of distinet variables is equal to $2^{(2^n)}$. In the case of n fuzzy variables, the number of fuzzy functions constructed in an arbitrary fashion with these n variables and the operations $\wedge$, v , and $^-$ is likewise finite; we shall prove this later.

Remark: Any v operation may be replaced by a $\wedge$ operation and vice versa. In fact

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = \text{MIN}\left(\underset{\sim}{a},\ \underset{\sim}{b}\right)$$

$$= 1 - \text{MAX}\left(\overline{\underset{\sim}{a}}, \overline{\underset{\sim}{b}}\right)$$

$$= \overline{\underset{\sim}{\overline{a}} \vee \underset{\sim}{\overline{b}}}\ .$$

This is another way of presenting (31.25). one may do the same for (31.26).

Thus it is sufficient to use either the operators ∧ and – or the operators v and – in order to represent any function of fuzzy variables involving the symbols ∧ , v , and -, but the notation is then very cumbersome.

Recall that in a Boolean algebra a single operator suffices to represent an arbitrary Boolean function.

Consider the Shaffer operator

$$a \mid b = \overline{a\ .\ b}$$

$$= \overline{a} \dotplus \overline{b}\ .$$

because

$$a \dotplus b = \overline{a} \mid \overline{b} = (a \mid a) \mid (b \mid b)$$

$$a\ .\ b = \overline{a \mid b} = (a \mid b) \mid (a \mid b)$$

$$\overline{a} = a \mid a\ .$$

Consider the Peirce operator

$$a \mid b = \overline{a \dotplus b}$$

$$= \overline{a}\ .\ \overline{b}\ .$$

because

$$a \dotplus b = \overline{a \mid b} = (a \mid b) \mid (a \mid b)\ .$$

$$a\ .\ b = \overline{a \mid b} = (a \mid a) \mid (a \mid b)\ .$$

$$\overline{a} = a \mid a\ .$$

One may pass from a Boolean expression using the Peirce operator to an expression involving the Sheffer operator and vice versa:

$$a \mid b = \overline{a}\ .\ \overline{b} = \overline{\overline{a} \mid \overline{b}} = \overline{(a \mid a) \mid (a \mid b)}$$

$$= ((a \mid a) \mid (b \mid b)) \mid ((a \mid a) \mid (b \mid b))\ .$$

CHARACTERISTIC FUNCTION OF A FUZZY SUBSET

$$a \mid b = \overline{a} \dotplus \overline{b} = \overline{\overline{a} \mid \overline{b}} = \overline{(a \mid b) \mid (a \mid b)}$$

$$((a \mid b) \mid (a \mid b))((a \mid b) \mid (a \mid b)$$

The difficulties in writing appear rapidly, so that one practically sets aside the use of such operators: but one may construct electronic circuits with a single technology and this may be useful in certain cases.

For the case of fuzzy variables, we define the operators

$$\underset{\sim}{a} \mid \underset{\sim}{b} = \overline{\underset{\sim}{a} \wedge \underset{\sim}{b}}$$

$$= \overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}}$$

$$\underset{\sim}{a} \mid \underset{\sim}{b} = \overline{\underset{\sim}{a} \vee \underset{\sim}{b}}$$

$$= \underset{\sim}{a} \wedge \underset{\sim}{b}$$

Any function of fuzzy variables may be written with the aid of only one of these operators. For

(1) $$\underset{\sim}{a} \vee \underset{\sim}{b} = \overline{\underset{\sim}{a}} \mid \overline{\underset{\sim}{b}} = \Big((\underset{\sim}{a} \mid \underset{\sim}{a}) \mid (b \mid b)\Big) .$$

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = \overline{\underset{\sim}{a} \mid \underset{\sim}{b}} = \Big((\underset{\sim}{a} \mid \underset{\sim}{b}) \mid (a \mid b)\Big) .$$

$$\overline{\underset{\sim}{a}} = \underset{\sim}{a} \mid \underset{\sim}{b} .$$

(2) $$\underset{\sim}{a} \vee \underset{\sim}{b} = \overline{\underset{\sim}{a}} \mid \overline{\underset{\sim}{b}} = \Big((\underset{\sim}{a} \mid \underset{\sim}{a}) \mid (a \mid b)\Big) .$$

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = \overline{\underset{\sim}{a}} \mid \overline{\underset{\sim}{b}} = \Big((\underset{\sim}{a} \mid \underset{\sim}{a}) \mid (b \mid b)\Big) .$$

$$\overline{\underset{\sim}{a}} = \underset{\sim}{a} \mid \underset{\sim}{a} .$$

And one may pass from peirce to Sheffer and vice versa using formulas (32.57) and 32.58) above.

As an example we see how to write a function of fuzzy variables that is not too complicated using the Sheffer operator:

$$\underset{\sim}{f}\ (\underset{\sim}{a},\ \underset{\sim}{b}\ ,\ \underset{\sim}{c}\ ) = \overline{\underset{\sim}{a}} \wedge \underset{\sim}{b} \vee \overline{\underset{\sim}{c}}$$

$$=(\underset{\sim}{a} \mid \underset{\sim}{a}) \wedge \Big((\underset{\sim}{b} \mid \underset{\sim}{b})\, \overline{\underset{\sim}{c}} \mid \overline{\underset{\sim}{c}}\,\Big)$$

$$= (\underset{\sim}{a} \mid \underset{\sim}{a}) \wedge \Bigg(\Big((\underset{\sim}{b} \mid \underset{\sim}{b}) \mid \Big(\underset{\sim}{c} \Big| \underset{\sim}{c}\,)\mid (\underset{\sim}{c} \mid \underset{\sim}{c}\,)\Big)\Bigg)$$

$$= (\underset{\sim}{a} \mid \underset{\sim}{a}) \mid \Bigg(\Big((\underset{\sim}{b} \mid \underset{\sim}{b}) \mid \Big(\underset{\sim}{c} \Big| \underset{\sim}{c}\,)\mid (\underset{\sim}{c} \mid \underset{\sim}{c}\,)\Big)\Bigg)\Bigg((\underset{\sim}{a} \mid \underset{\sim}{a}) \mid \Big((\underset{\sim}{b} \mid \underset{\sim}{b}) \mid \Big(\underset{\sim}{c} \Big| \underset{\sim}{c}\,)\mid (\underset{\sim}{c} \mid \underset{\sim}{c}\,)\Big)\Bigg)$$

This is a horribly complicated expression to express a function so simple as

$\underset{\sim}{\overline{a}} \wedge (\underset{\sim}{\overline{b}} \vee \underset{\sim}{c})$.

Table of values of a function of fuzzy variables. In order to study Boolean binary functions one may use what is called a truth table, where one assigns to the binary variables all possible values and obtains thanks to this the values of the function. Such a truth table would not make sense for functions of fuzzy variables, but one may construct tables of a different nature that play a similar role.

In order to study a function of one fuzzy variable $\underset{\sim}{a}$ we shall examine its value in the following two cases:

$\underset{\sim}{a} \leq \underset{\sim}{\overline{a}}$.

$\underset{\sim}{\overline{a}} \leq \underset{\sim}{a}$.

In order to study a function of two variables $\underset{\sim}{a}$ and $\underset{\sim}{b}$ we shall examine its value in the following eight cases.

$\underset{\sim}{a} \leq \underset{\sim}{b} \leq \underset{\sim}{\overline{b}} \leq \underset{\sim}{\overline{a}}$.

$\underset{\sim}{a} \leq \underset{\sim}{\overline{b}} \leq \underset{\sim}{b} \leq \underset{\sim}{\overline{a}}$.

$\underset{\sim}{\overline{a}} \leq \underset{\sim}{b} \leq \underset{\sim}{\overline{b}} \leq \underset{\sim}{a}$.

$\underset{\sim}{a} \leq \underset{\sim}{\overline{b}} \leq \underset{\sim}{b} \leq \underset{\sim}{a}$.

$\underset{\sim}{b} \leq \underset{\sim}{a} \leq \underset{\sim}{\overline{a}} \leq \underset{\sim}{\overline{b}}$.

$\underset{\sim}{b} \leq \underset{\sim}{\overline{a}} \leq \underset{\sim}{a} \leq \underset{\sim}{\overline{b}}$.

$\underset{\sim}{b} \leq \underset{\sim}{a} \leq \underset{\sim}{\overline{a}} \leq \underset{\sim}{b}$.

$\underset{\sim}{\overline{b}} \leq \underset{\sim}{\overline{a}} \leq \underset{\sim}{a} \leq \underset{\sim}{b}$.

In order to study a function of the three variables $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ one considers the 48 cases below, presented without the $<$ sign and without the symbol $\sim$ in order to save space:

$a\ b\ c\ \overline{c}\ \overline{b}\ \overline{a}$.	$\overline{a}\ b\ \overline{c}\ c\ \overline{b}\ a$.	$\overline{a}\ c\ b\ \overline{b}\ \overline{c}\ a$.
$a\ b\ \overline{c}\ c\ \overline{b}\ \overline{a}$.		$\overline{a}\ c\ \overline{b}\ c\ \overline{a}\ a$.
$a\ \overline{b}\ c\ \overline{c}\ b\ \overline{a}$.	$a\ c\ b\ \overline{b}\ \overline{c}\ \overline{a}$.	
$a\ \overline{b}\ \overline{c}\ c\ b\ \overline{a}$.	$a\ c\ \overline{b}\ b\ \overline{c}\ \overline{a}$.	$b\ a\ c\ \overline{c}\ \overline{a}\ \overline{b}$.
$\overline{a}\ b\ c\ \overline{c}\ \overline{b}\ a$.	$a\ \overline{c}\ b\ \overline{b}\ c\ \overline{a}$.	$b\ a\ \overline{c}\ c\ \overline{a}\ \overline{b}$.
	$a\ \overline{c}\ \overline{b}\ b\ c\ \overline{a}$.	$b\ \overline{a}\ c\ \overline{c}\ a\ \overline{b}$.

$b\ \bar{a}\ \bar{c}\ \ c\ a\ \ \bar{b}$. $\bar{b}\ \ a\ c\ \bar{c}\ \ \bar{a}\ b$. $\bar{b}\ \ a\ \ \ \bar{c}\ c\ \ \bar{a}\ b$.

1 suggest calling these enumeration procedures entipatindromes since one forms palindrome sequences and superimposes complementation on these in an antisymmetric fashion. (A palindrome is a world of phrase that is identical to itself when one reads it from the end to the beginning: “Able was | ere | saw Elba.”)

$\overline{a}\ b\ c\ \overline{c}\ \overline{b}\ a$

$\overline{a}\ b\ \overline{c}\ c\ b\ \ a$

$b\ c\ a\ \overline{a}\ c\ b.$

$b\,c\ \overline{a}\ a\ \overline{c}\ \ \overline{b}\ .$

$b\ \overline{c}\ a\ \overline{a}\ c\ \overline{b}\,.$

$b\ \overline{c}\ \overline{a}$ a $c\,\overline{b}\ .$

$\overline{b}\ \overline{c\ a\ a\ c}\ b\ .$

$\overline{b}\ c\ \overline{a}\ a\ \overline{c}\ b\ .$

$\overline{b}\ c\ a\ a\ c\ b.$

$b\ c\ a\ a\ c\ b$

$\overline{a}\ \overline{c}\ b\ \overline{b}\ c\ \ a\ .$

$\overline{a}\ c\ \overline{b}\ b\ \ c\ a.$

$c\,a\ b\ \overline{b}\ \ \overline{a}\ \ \overline{c}\ .\quad c\,a\ \overline{b}\ b\ \ \overline{a}\ \ \overline{c}.c\ \overline{a}\ b\ \ \overline{b}\ a\ \ \overline{c}\ .$

$c\ \overline{a}\ \overline{b}\ \ b\ a\ \ \overline{c}\ .$

$c\ \overline{a}\ b\ \ \overline{b}\ a\ \ \overline{c}\ .$

$\overline{c}\ \ a\ b\ \overline{b}\ \ \overline{a}\ c.$

$c\ \overline{a}\ b$ b $a\ c.$

$\overline{c}\ \overline{a}\ \overline{b}\ \ b\ a\ c.$

$\overline{b}\ \ \overline{a}\ c\ \overline{c}\ a\ b$

$\overline{b}\ \ \overline{a}\ \overline{c}\ c\,a\ \ b$

$c\ b\ a\ \overline{a}\ b\ \overline{c}.$

$\overline{c}\ b\ \overline{a}\ a\,\overline{b}\ \overline{c}.$

c $\overline{b}\ a\ \overline{a}\ b\ \overline{c}$

c $\overline{b}\ \overline{a}\ a\,b\,\overline{c}$

$c\ b\ a\ a\,\overline{b}\ c$

$\overline{c}\ b\ \overline{a}\,a\,b\ c.$

$\overline{c}\ \ \overline{b}\,a\,\overline{a}\ b\,c.$

c $\overline{b}\ \overline{a}\ a\,b\,c$

In order to study a function of *n* variables, one considers

$$p_n\ \ 2^n \text{ where } p_n = \mathrm{n} = \mathrm{n}(\mathrm{n}-1)$$

By examining (32.68) – (32.70) one may establish the effect of antisymmetry owing to

$$\text{if} \quad \underset{\sim}{v} \leq \underset{\sim}{v} \qquad \text{then} \qquad \underset{\sim}{\overline{v}} \leq \underset{\sim}{\overline{v}}$$

To enumerate all possible cases without omission or repetition. one uses a lexicographic procedure. Establish, for example, the correspondence

1	$a.$
2	$a.$
3	$b.$
4	$\overline{b}.$

then one has the correspondences

1 3	$a \ b$	from which	$a \ b \ \overline{b} \ \overline{a}$
1 4	$a \ b$	from which	$a \ \overline{b} \ b \, \overline{a}$
2 3	$\overline{a} \ b$	from which	$\overline{a} \ b \ \overline{b} \ a$
1 3	$a \ b$	from which	$\overline{a} \ \overline{b} \ b \ a$
1 3	$a \ b$	from which	$b \ a \ \overline{a} \ \overline{b}$
1 3	$a \ b$	from which	$b \ \overline{a} \ a \ \overline{b}$
1 3	$a \ b$	from which	$\overline{b} \ a \ \overline{a} \ b$
1 3	$a \ b$	from which	$\overline{b} \ \overline{a} \ a \ b$

other procedures may easily be imagined.

We consider an example, Enumerate the values of the function

$$\underset{\sim}{f} \left(\underset{\sim}{a} , \underset{\sim}{b} \right) = \left(\underset{\sim}{a} \wedge \underset{\sim}{\overline{a}} \right) \ \mathrm{v} \left(\underset{\sim}{\overline{a}} \wedge \underset{\sim}{b} \ \wedge \underset{\sim}{\overline{b}} \right)$$

the result is supplied in the table in Figure 31.1

$\leq \ \leq \ \leq$				$\underset{\sim}{a} \wedge \underset{\sim}{\overline{a}}$	$\underset{\sim}{\overline{a}} \wedge \underset{\sim}{b} \ \wedge \underset{\sim}{\overline{b}}$	$\left(\underset{\sim}{a} \wedge \underset{\sim}{\overline{a}} \right) \ \mathrm{v} \left(\underset{\sim}{\overline{a}} \wedge \underset{\sim}{b} \ \wedge \underset{\sim}{\overline{b}} \right)$
$\underset{\sim}{a}$	$\underset{\sim}{b}$	$\underset{\sim}{\overline{b}}$	$\underset{\sim}{\overline{a}}$	$\underset{\sim}{a}$	$\underset{\sim}{b}$	$\underset{\sim}{b}$
$\underset{\sim}{a}$	$\underset{\sim}{b}$	$\underset{\sim}{b}$	$\underset{\sim}{\overline{a}}$	$\underset{\sim}{a}$	$\underset{\sim}{\overline{b}}$	$\underset{\sim}{\overline{b}}$

$\underset{\sim}{b}$	$\underset{\sim}{b}$	$\underset{\sim}{\bar{b}}$	$\underset{\sim}{a}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{\bar{a}}$
$\underset{\sim}{\bar{a}}$	$\underset{\sim}{\bar{b}}$	$\underset{\sim}{b}$	$\underset{\sim}{a}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{\bar{a}}$
$\underset{\sim}{b}$	$\underset{\sim}{a}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{\bar{b}}$	$\underset{\sim}{a}$	$\underset{\sim}{b}$	$\underset{\sim}{\bar{a}}$
$\underset{\sim}{b}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{a}$	$\underset{\sim}{\bar{b}}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{b}$	$\underset{\sim}{\bar{a}}$
$\underset{\sim}{\bar{b}}$	$\underset{\sim}{a}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{b}$	$\underset{\sim}{a}$	$\underset{\sim}{\bar{b}}$	$\underset{\sim}{a}$
$\underset{\sim}{\bar{b}}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{a}$	$\underset{\sim}{b}$	$\underset{\sim}{\bar{a}}$	$\underset{\sim}{\bar{b}}$	$\underset{\sim}{\bar{a}}$

Equality of two functions of fuzzy variables. We shall say that two functions of fuzzy variables $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2$ are equal (one also says identical) if they produce the same table of values through enumeration of all possible cases.

Mixed operations. The variables $\underset{\sim}{a}$, $\underset{\sim}{b}$ ….. ∈ = [0,1]. may be submitted to0 operations other than ∧, v and — in order to form what will be called mixed functions of fuzzy variables.

A among such operations we shall include

Product $\underset{\sim}{a} \,.\, \underset{\sim}{b}$ where one can easily verify

$$\underset{\sim}{a} \in = [0,1],\ \underset{\sim}{b} \in = [0,1] - \underset{\sim}{a} \,.\, \underset{\sim}{b} \in = [0,1].$$

Sum $$\underset{\sim}{a} \hat{+} \underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{a} \,.\, \underset{\sim}{b}\,.$$

where the above property still holds.

Thus, the function

$$\underset{\sim}{f}\,(\underset{\sim}{a},\ \underset{\sim}{b},\underset{\sim}{c}) = \left(\underset{\sim}{a} \hat{+} \underset{\sim}{b}\right) \wedge \left(\underset{\sim}{b} \hat{+} \underset{\sim}{\bar{c}}\right) \wedge \underset{\sim}{a} \wedge \underset{\sim}{c}\,.$$

is a mixed function.

Important remark. With the aid of a table of enumeration one may define for n variables.

N = $(2n)^{(n.2n)}$

distinet functions + : thus

$n = 1$ N = $(2.1)^2 = 2^2 = 4$.

$n = 2$ N = $(2.2)^{2.2^2} = 4^2 = 65.536$.

$n = 3$ N = $(2.2)^{6.2^3} = 6^{45}$.

$n = 4$ N = $(2.4)^{24.2^4} = 8^{384}$ ………… etc.

among all these functions, a considerably smaller number are formed by the functions of fuzzy variables expressible with the aid of the operations $\wedge$ and v on the variables $\underset{\sim}{a}, \underset{\sim}{b}$ ……. and $\underset{\sim}{a}, \underset{\sim}{b}$ …….

Convention 'Unless otherwise noted, we shall call an analytic function of fuzzy variables, designated by $\underset{\sim}{f}$. any function of the variables $\underset{\sim}{a} . \underset{\sim}{b}$ ……. that may be expressed using only the operations $\wedge$ and v : the variables may occur either in their direct form or as their 1's complement, that is $\underset{\sim}{a} . \underset{\sim}{b}$ ……

In order to simplify language, already rather cumbersome, analytic functions of fuzzy variables will be called functions of fuzzy variables when this will not introduce error or confusion.

32. POLYNOMIAL FORMS

Given the double distributivity expressed by (32.18) and (32.19), any function $\underset{\sim}{f}$ $(\underset{\sim}{a} . \underset{\sim}{b}$ …..) may be expressed in a polynomial form with respect to $\wedge$ or with respect to v.

To begin, we consider an example. Let

$$\overline{\underset{\sim}{f}}\,(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) = (\overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}}) \vee (\underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}})$$

This function is presented in a polynomial form with respect to v (two monomials in $\wedge$ connected by v). We may transform this into a polynomial form with respect to $\wedge$ by using (31.19) ; it becomes

$= \underset{\sim}{f}\,(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) = (\overline{\underset{\sim}{a}} \vee \underset{\sim}{a}) \wedge (\overline{\underset{\sim}{a}} \vee \underset{\sim}{a} \wedge \underset{\sim}{b}) \wedge (\overline{\underset{\sim}{a}} \vee \overline{\underset{\sim}{c}}) \wedge (\overline{\underset{\sim}{b}} \vee \underset{\sim}{a}) \wedge (\overline{\underset{\sim}{b}} \vee \underset{\sim}{b}) \wedge (\overline{\underset{\sim}{b}} \vee \overline{\underset{\sim}{c}})$

we see another example. Let

This is easy to prove. For a variables a, b, ….. l. one must have n1 permulations. But, in each permutation one must have a or $\overline{a}$, b or $\overline{b}$ ….. 1 or $\overline{1}$ thus 2^n times more permutations;

this give n1. 2^2 as the number of rows in a table such as that in Figure 31.1. Each row may take a value among these 2_n variables and their respective complements; thus one may define $(2\mathrm{n})^{(n!.2^n)}$ distinct functions with such tables.

$$\underset{\sim}{f}\,(\underset{\sim}{a}\,,\underset{\sim}{b}\,,\underset{\sim}{c}) \;=\; (\underset{\sim}{a} \vee \underset{\sim}{b}) \wedge \underset{\sim}{c} \;\wedge (\underset{\sim}{\overline{a}} \vee \underset{\sim}{b} \vee \underset{\sim}{c}\,) = (\underset{\sim}{a} \vee \underset{\sim}{\overline{b}}) \wedge \underset{\sim}{c}$$

$$by\ absorption\ of\ the\ third\ term\ by\ the\ second.\ Developing\ this\ using\ (31.18).$$

$$we\ have$$

$$\underset{\sim}{f}\,(\underset{\sim}{a}\,,\underset{\sim}{b}\,,\underset{\sim}{c}) \;= (\underset{\sim}{a} \wedge \underset{\sim}{c}\,) \vee (\,\underset{\sim}{\overline{b}} \wedge \underset{\sim}{c}\,)$$

which gives a polynomial form with respect to ∨ ; whereas $(\underset{\sim}{a} \vee \underset{\sim}{b}) \wedge \underset{\sim}{c}$ is the corresponding polynomial form with respect to ∧.

In the case of Boolean functions, in order to show that two functions $\underset{\sim}{f}$ and $\underset{\sim}{f}^{\cdot}$ are identical, it suffices to check that they lead to the same truth table or that their disjunctive or conjunctive canonical forms are respectively the same. Concerning functions of fuzzy variables, one may define a similar but less strong notion.

Maximal monomial. Let $\underset{\sim}{f}\,\left(\underset{\sim}{a}\,,\underset{\sim}{b}\ \ldots\right)$ be expressed in a polynomial form with respect to ∧. A monomial of this polynomial form will be said to be maximal (one also says principal monomial) if it is absorbed by no other monomial of this polynomial form (a corresponding definition is made for a monomial in a polynomial form with respect to ∨).

Reduced polynomial form. Any polynomial form with respect to ∨ that does not contain a maximal monomial in ∧ will be said to be a *reduced polynomial form with respect to* ∨. A symmetric definition, by permuting ∨ and ∧, will define a *reduced polynomial form with respect to* ∧.

An analytic function $\underset{\sim}{f}\,\left(\underset{\sim}{a}\,,\underset{\sim}{b}\ \ldots\right)$ may correspond to several reduced polynomial forms. We shall see an example. The two reduced polynomial forms

$$\underset{\sim}{f}\left(\underset{\sim}{a}\,,\underset{\sim}{b}\right) = \left(\underset{\sim}{a} \wedge \underset{\sim}{\overline{a}}\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{\overline{a}}\right)$$

$$\underset{\sim}{f}\left(\underset{\sim}{a}\,,\underset{\sim}{b}\right) = \left(\underset{\sim}{a} \wedge \underset{\sim}{b}\right) \vee \left(\underset{\sim}{a} \wedge \underset{\sim}{\overline{a}}\right)$$

correspond to the same analystic function, as one may verify by antipalindrome enumeration, as has been done, for example, in Figure 32.1.

For any analytic function, there exists at least one reduced polynomial form with respect to ∨ and at least one reduced polynomial form with respect to ∧. One may pass from one to the other by developing with respect to ∧ (respectively, with respect to ∨) and effecting the absorption of nonmaximal monomials. In Appendix C of Volume II we shall treat these notions in more detail.

Example: The function.

$$\underset{\sim}{f}\left(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}\right) = \left(\underset{\sim}{\overline{a}} \wedge \underset{\sim}{b} \wedge \underset{\sim}{\overline{c}}\right) \vee \left(\underset{\sim}{\overline{b}} \wedge \underset{\sim}{c}\right)$$

is presented in a reduced polynomial form with respect to ∨.

Its reduced polynomial form with respect to ∧ is

$$\underset{\sim}{f}\left(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}\right) = \left(\underset{\sim}{\overline{a}} \vee \underset{\sim}{\overline{b}}\right) \wedge \left(\underset{\sim}{\overline{a}} \vee \underset{\sim}{c}\right) \wedge \left(\underset{\sim}{b} \vee \underset{\sim}{\overline{b}}\right) \wedge \left(\underset{\sim}{b} \vee \underset{\sim}{c}\right) \wedge \left(\underset{\sim}{c} \vee \underset{\sim}{\overline{c}}\right) \wedge \left(\underset{\sim}{\overline{b}} \vee \underset{\sim}{\overline{c}}\right).$$

Identity of two functions of fuzzy variables. A sufficient condition for two functions of fuzzy variables to be identical is that one can bring them to the same reduced polynomial form in ∧ (respectively, in ∨). A necessary and sufficient condition is that one obtains the same table of values for the functions.

Theorem. The number of distinct reduced polynomial forms in *n* variables is finite and is a superior bound for the number of distinct analytic functions of *n* fuzzy variables.

As may be seen in the enumeration that follows, these reduced polynomials forms are enumerated as the elements of a distributive free lattice with $2n$ generators and are enumerated in the same fashion. Thus, for $n = 1$. there are 4 distinct forms. for $n = 2$, there are 166; for $n = 3$, there are 7,828,532…; but this number of distinct forms always remains finite because the number of elements of a distributive free lattice with $2n$ generators is always finite in *n* is finite.

The enumeration of all reduced forms of n fuzzy variables does not seem to be an easy problem.

For on: variable. it is trivial. One has

$$\underset{\sim}{a} \quad . \quad \underset{\sim}{\overline{a}} \quad . \quad \underset{\sim}{a} \wedge \underset{\sim}{\overline{a}} \quad . \quad \underset{\sim}{a} \wedge \underset{\sim}{\overline{a}} \quad .$$

that is, four reduced forms. Note well that $\underset{\sim}{a} \wedge \underset{\sim}{b}$, for example, is to be distinguished from $\underset{\sim}{a}$ since

$$\underset{\sim}{a} \wedge \underset{\sim}{\overline{a}} = \underset{\sim}{a} \quad \text{if} \quad \underset{\sim}{a} \leq \underset{\sim}{\overline{a}} \text{ and } \quad \underset{\sim}{a} \wedge \underset{\sim}{\overline{a}} = \underset{\sim}{\overline{a}} \quad \text{if} \quad \underset{\sim}{\overline{a}} \leq \underset{\sim}{a}.$$

For two variables, it is already no longer simple, and is in fact very complicated.

We use, for example, reduced polynomial forms in ∨ (monomials in ∧). We know that to each form in ∨ there corresponds a form in ∧ and vice versa (because of the two theorems of De Morgan).

We then see the enumeration + of all possible distinct reduced polynomials forms in ∨ $\underset{\sim}{f}\left(\underset{\sim}{a}, \underset{\sim}{b}\right)$:

(1) $\underset{\sim}{f}\left(\underset{\sim}{a}, \underset{\sim}{b}\right)$ containing one monomial:

1	a (1)	$1 \wedge 2$	$a \wedge \bar{a}$ (5)	$1 \wedge 2 \wedge 3$	$a \wedge \bar{a} \wedge b$ (11)		
2	$\bar{a}$ (2)	$1 \wedge 3$	$a \wedge b$ (6)	$1 \wedge 2 \wedge 4$	$a \wedge \bar{a} \wedge \bar{b}$ (12)		
3	b (3)	$1 \wedge 4$	$a \wedge \bar{b}$ (7)	$1 \wedge 3 \wedge 4$	$a \wedge b \wedge \bar{b}$ (13)		
4	$\bar{b}$ (4)	$2 \wedge 3$	$\bar{a} \wedge b$ (8)	$2 \wedge 3 \wedge 4$	$\bar{a} \wedge b \wedge \bar{b}$ (14)		
		$2 \wedge 4$	$\bar{a} \wedge \bar{b}$ (9)				
		$3 \wedge 4$	$b \wedge \bar{b}$ (10)			$1 \wedge 2 \wedge 3 \wedge 4$	$a \wedge \bar{a} \wedge b \wedge \bar{b}$ (15)
4		6		4		1	

(2) $\underset{\sim}{f}\left(\underset{\sim}{a}, \underset{\sim}{b}\right)$ containing two monomials then :

$1 \vee 2$	$a \vee \bar{a}$ (16)
$1 \vee 3$	$a \vee b$ (17)
$1 \vee 4$	$a \vee \bar{b}$ (18)
$2 \vee 3$	$\bar{a} \vee b$ (19)
$2 \vee 4$	$\bar{a} \vee \bar{b}$ (20)
$3 \vee 4$	$b \vee \bar{b}$ (21)

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$1 \vee (2 \wedge 3)$	$a \vee (\bar{a} \wedge b)$ (22)
$1 \vee (2 \wedge 4)$	$a \vee (\bar{a} \wedge \bar{b})$ (23)
$1 \vee (3 \wedge 4)$	$a \vee (b \wedge \bar{b})$ (24)
$2 \vee (1 \wedge 3)$	$\bar{a} \vee (a \wedge b)$ (25)
$2 \vee (1 \wedge 4)$	$\bar{a} \vee (a \wedge \bar{b})$ (26)
$2 \vee (3 \wedge 4)$	$\bar{a} \vee (b \wedge \bar{b})$ (27)
$3 \vee (1 \wedge 2)$	$b \vee (a \wedge \bar{a})$ (28)
$3 \vee (1 \wedge 4)$	$b \vee (a \wedge \bar{b})$ (29)
$3 \vee (2 \wedge 4)$	$b \vee (\bar{a} \wedge \bar{b})$ (30)
$4 \vee (1 \wedge 2)$	$\bar{b} \vee (a \wedge \bar{a})$ (31)
$4 \vee (1 \wedge 3)$	$\bar{b} \vee (a \wedge b)$ (32)
$4 \vee (2 \wedge 3)$	$\bar{b} \vee (\bar{a} \wedge b)$ (33)

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$1 \vee (2 \wedge 3 \wedge 4)$	$a \vee (\bar{a} \wedge b \wedge b)$ (34)
$2 \vee (1 \wedge 3 \wedge 4)$	$\bar{a} \vee (a \wedge b \wedge \bar{b})$ (35)
$3 \vee (1 \wedge 2 \wedge 4)$	$b \vee (a \wedge \bar{a} \wedge \bar{b})$ (36)
$4 \vee (1 \wedge 2 \wedge 3)$	$\bar{b} \vee (a \wedge \bar{a} \wedge b)$ (37)

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(33.12)

$(1 \wedge 2) \vee (1 \wedge 3)$	$(a \wedge \bar{a}) \vee (a \wedge b)$ (38)
$(1 \wedge 2) \vee (1 \wedge 4)$	$(a \wedge \bar{a}) \vee (a \wedge \bar{b})$ (39)
$(1 \wedge 2) \vee (2 \wedge 3)$	$(a \wedge \bar{a}) \vee (\bar{a} \wedge b)$ (40)
$(1 \wedge 2) \vee (2 \wedge 4)$	$(a \wedge \bar{a}) \vee (\bar{a} \wedge \bar{b})$ (41)
$(1 \wedge 2) \vee (3 \wedge 4)$	$(a \wedge \bar{a}) \vee (b \wedge \bar{b})$ (42)
$(1 \wedge 3) \vee (1 \wedge 4)$	$(a \wedge b) \vee (a \wedge \bar{b})$ (43)
$(1 \wedge 3) \vee (2 \wedge 3)$	$(a \wedge b) \vee (\bar{a} \wedge b)$ (44)
$(1 \wedge 3) \vee (2 \wedge 4)$	$(a \wedge b) \vee (\bar{a} \wedge \bar{b})$ (45)
$(1 \wedge 3) \vee (3 \wedge 4)$	$(a \wedge b) \vee (b \wedge \bar{b})$ (46)
$(1 \wedge 4) \vee (2 \wedge 3)$	$(a \wedge \bar{b}) \vee (\bar{a} \wedge b)$ (47)
$(1 \wedge 4) \vee (2 \wedge 4)$	$(a \wedge \bar{b}) \vee (\bar{a} \wedge \bar{b})$ (48)
$(1 \wedge 4) \vee (3 \wedge 4)$	$(a \wedge \bar{b}) \vee (b \wedge \bar{b})$ (49)
$(2 \wedge 3) \vee (2 \wedge 4)$	$(\bar{a} \wedge b) \vee (\bar{a} \wedge \bar{b})$ (50)
$(2 \wedge 3) \vee (3 \wedge 4)$	$(\bar{a} \wedge b) \vee (b \wedge \bar{b})$ (51)
$(2 \wedge 4) \vee (3 \wedge 4)$	$(\bar{a} \wedge \bar{b}) \vee (b \wedge \bar{b})$ (52)

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$(1 \wedge 2) \vee (1 \wedge 3 \wedge 4)$	$(a \wedge \bar{a}) \vee (a \wedge b \wedge \bar{b})$ (53)
$(1 \wedge 2) \vee (2 \wedge 3 \wedge 4)$	$(a \wedge \bar{a}) \vee (\bar{a} \wedge b \wedge \bar{b})$ (54)
$(1 \wedge 3) \vee (1 \wedge 2 \wedge 4)$	$(a \wedge b) \vee (a \wedge \bar{a} \wedge \bar{b})$ (55)
$(1 \wedge 3) \vee (2 \wedge 3 \wedge 4)$	$(a \wedge b) \vee (\bar{a} \wedge b \wedge \bar{b})$ (56)
$(1 \wedge 4) \vee (1 \wedge 2 \wedge 3)$	$(a \wedge \bar{b}) \vee (a \wedge \bar{a} \wedge b)$ (57)
$(1 \wedge 4) \vee (2 \wedge 3 \wedge 4)$	$(a \wedge \bar{b}) \vee (\bar{a} \wedge b \wedge \bar{b})$ (58)
$(2 \wedge 3) \vee (1 \wedge 2 \wedge 4)$	$(\bar{a} \wedge b) \vee (a \wedge \bar{a} \wedge \bar{b})$ (59)
$(2 \wedge 3) \vee (1 \wedge 3 \wedge 4)$	$(\bar{a} \wedge b) \vee (a \wedge b \wedge \bar{b})$ (60)
$(2 \wedge 4) \vee (1 \wedge 2 \wedge 3)$	$(\bar{a} \wedge \bar{b}) \vee (a \wedge \bar{a} \wedge b)$ (61)
$(2 \wedge 4) \vee (1 \wedge 3 \wedge 4)$	$(\bar{a} \wedge \bar{b}) \vee (a \wedge b \wedge \bar{b})$ (62)
$(3 \wedge 4) \vee (1 \wedge 2 \wedge 3)$	$(b \wedge \bar{b}) \vee (a \wedge \bar{a} \wedge b)$ (63)
$(3 \wedge 4) \vee (1 \wedge 2 \wedge 4)$	$(b \wedge \bar{b}) \vee (a \wedge \bar{a} \wedge \bar{b})$ (64)

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$(1 \wedge 2 \wedge 3) \vee (1 \wedge 2 \wedge 4)$	$(a \wedge \bar{a} \wedge b) \vee (a \wedge \bar{a} \wedge \bar{b})$ (65)
$(1 \wedge 2 \wedge 3) \vee (1 \wedge 3 \wedge 4)$	$(a \wedge \bar{a} \wedge b) \vee (a \wedge b \wedge \bar{b})$ (66)
$(1 \wedge 2 \wedge 3) \vee (2 \wedge 3 \wedge 4)$	$(a \wedge \bar{a} \wedge b) \vee (\bar{a} \wedge b \wedge \bar{b})$ (67)
$(1 \wedge 2 \wedge 4) \vee (1 \wedge 3 \wedge 4)$	$(a \wedge \bar{a} \wedge \bar{b}) \vee (a \wedge b \wedge \bar{b})$ (68)
$(1 \wedge 2 \wedge 4) \vee (2 \wedge 3 \wedge 4)$	$(a \wedge \bar{a} \wedge \bar{b}) \vee (\bar{a} \wedge b \wedge \bar{b})$ (69)
$(1 \wedge 3 \wedge 4) \vee (2 \wedge 3 \wedge 4)$	$(a \wedge b \wedge \bar{b}) \vee (\bar{a} \wedge b \wedge \bar{b})$ (70)

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33. ANALYSIS OF A FUNCTION OF FUZZY VARIABLES. METHOD OF MARINOS

We decompose M = $[0,1]$ in to m joined intervals, closed on the left and open on the right, except the last:

$$1_1 = [\ \alpha_0 = 0, \alpha_1\ ] \quad . \quad 1_2 = \alpha_1, \alpha_2 1 \ldots\ldots 1_m = [\alpha_{m-1}, \alpha_m = 1].$$

where

$$M = ([\ \alpha_0 = 0, \alpha_1\ ]) \cup ([\alpha_1, \alpha_2\ ]) \cup \ldots\ldots \cup ([\alpha_{m-1}, \alpha_m = 1\ ]).$$

we then seek conditions so that a function of n fuzzy variables

$$\underset{\sim}{f}\left(\underset{\sim}{a_1}, \underset{\sim}{a_2} \ldots\ldots\ldots \underset{\sim}{a_n}\right) \quad . \quad \underset{\sim}{a_1} \in [0,1] \quad . \quad 1 = 1, 2, \ldots\ldots n \ .$$

will belong to an interval l_k.

Example - 1. We begin with an example.

$$\text{Let } \underset{\sim}{f}\left(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}\right) = \left(\overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}}\right) \text{v} \left(\underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}}\right).$$

What conditions will give one

$$\underset{\sim}{f}\left(\underset{\sim}{a} . \underset{\sim}{b} . \underset{\sim}{c}\right) \in 1_A .$$

that is,

$$\alpha_{A-1} \leq \underset{\sim}{f}\left(\underset{\sim}{a} . \underset{\sim}{b} . \underset{\sim}{c}\right) < \alpha_A .$$

We examine (43.4). The member on the right is formed of two terms; thus it is necessary to take the largest. We begin with a first hypothesis.

Hypothesis 1: $\overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}} \geq \underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}}$

This implies $\alpha_{A-1} \leq \overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}} < \alpha_A$.

that is, explicitly

$$\alpha_{A-1} \leq \text{MIN}\left(\overline{\underset{\sim}{a}}, \overline{\underset{\sim}{b}}\right) < \alpha_A .$$

or again

$$\alpha_{A-1} \leq \text{MIN}\left(1 - \underset{\sim}{a} . 1 - \underset{\sim}{b}\right) < \alpha_A .$$

Since one may not place $\underset{\sim}{a}$ and $\underset{\sim}{b}$ arbitrarily with respect to one another, it is necessary that

$$1\text{-} \underset{\sim}{a} \geq \alpha_{A-1} \text{ and } 1 - \underset{\sim}{b} \geq \alpha_{A-1} .$$

and

$$1\text{-} \underset{\sim}{a} \leq \alpha_{A-1} \text{ or / and } 1 - \underset{\sim}{b} < \alpha_A .$$

This may be rewritten:

$$\underset{\sim}{a} \leq 1 - \alpha_{A-1} \text{ and } \underset{\sim}{b} \leq 1 - \alpha_{A-1}$$

and

$$\underset{\sim}{a} > 1 - \alpha_k \text{ or / and } 1 - \underset{\sim}{b} > 1 - \alpha_k$$

Hypothesis – II:

$$\overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}} < \underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}}$$

This implies

$$\alpha_{k-1} \leq \underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}} < \alpha_k .$$

explicitly that is,

$$\alpha_{k-1} \leq \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}, \overline{\underset{\sim}{c}}\right) < \alpha_k .$$

or again

$$\alpha_{k-1} \leq \text{MIN}\left(\underset{\sim}{a}, \underset{\sim}{b}, 1 - \underset{\sim}{c}\right) < \alpha_k .$$

Since we may not place $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ arbitrarily with respect to one another, first it is necessary that

$$\underset{\sim}{a} \geq \alpha_{k-1} \text{ and } 1 - \underset{\sim}{b} \geq \alpha_{k-1} \text{ and } 1 - \underset{\sim}{c} \geq \alpha_{k-1}$$

and

$$\underset{\sim}{a} \leq \alpha_k \text{ or / and } 1 - \underset{\sim}{b} < \alpha_k \text{ or / and } 1 - \underset{\sim}{c} < \alpha_k .$$

This may be rewritten:

$$\underset{\sim}{a} \geq \alpha_{k-1} \text{ and } \underset{\sim}{b} \geq \alpha_{k-1} \text{ and } \underset{\sim}{c} \leq 1 - \alpha_{k-1}$$

and

$$\underset{\sim}{a} < \alpha_k \text{ or / and } \underset{\sim}{b} < \alpha_k \text{ Or / and } \underset{\sim}{c} > 1 - \alpha_k$$

Finally, these results may be regrouped in the following fashion:

Property: I,

$$\left[\left(\underset{\sim}{a} \leq 1 - \alpha_{k-1}\right) \text{and} \left(\underset{\sim}{b} \leq 1 - \alpha_{k-1}\right)\right] \text{ or / and} \left[\left(\underset{\sim}{a} \geq \alpha_{k-1}\right) \text{and} \left(\underset{\sim}{b} \geq \alpha_{k-1}\right) \text{and} \left(\underset{\sim}{c} \leq 1 - \alpha_{k-1}\right)\right]$$

Property: II,

$$\left[\left(\underset{\sim}{a} > 1 - \alpha_k\right) \text{or / and} \left(\underset{\sim}{b} > 1 - \alpha_k\right)\right] \quad \text{and} \quad \left[\left(\underset{\sim}{a} < \alpha_k\right) \text{or / and} \left(\underset{\sim}{b} < \alpha_k\right) \text{or / and} \left(\underset{\sim}{c} > 1 - \alpha_k\right)\right]$$

As a sample of the calculation of $\underset{\sim}{f}\left(\underset{\sim}{a} . \underset{\sim}{b} . \underset{\sim}{c}\right)$ for particular numerical values, we suppose that

$$\underset{\sim}{a} = 0.55 \qquad \underset{\sim}{b} = 0.57 \quad . \quad \underset{\sim}{c} = 0.80 .$$

Then one has

$$\underset{\sim}{f}\left(\underset{\sim}{a} . \underset{\sim}{b} . \underset{\sim}{c}\right) = \underset{\sim}{f}\,(0.55 ; 0.57 ; 0.80)$$

$= \left(\overline{\underset{\sim}{a}} \wedge \overline{\underset{\sim}{b}}\right) \text{v} \left(\underset{\sim}{a} \wedge \underset{\sim}{b} \wedge \overline{\underset{\sim}{c}}\right)$ where $\underset{\sim}{a} = 0.55$; $\underset{\sim}{b} = 0.57$; $\underset{\sim}{c} = 0.80$

$= (0.45 \wedge 0.43) \text{ v } (0.55 \wedge 0.57 \wedge 0.20)$

$= 0.43 \text{ v } 0.20$

$= 0.43 .$

We now consider a complete numerical example.

Example 2 Let

$$\underset{\sim}{f}\left(\underset{\sim}{a}\,.\,\underset{\sim}{b}\,.\,\underset{\sim}{c}\right)=\left(\underset{\sim}{a}\ \wedge\ \underset{\sim}{\overline{b}}\right)\text{ v }\left(\underset{\sim}{\overline{a}}\ \wedge\ \underset{\sim}{c}\ \right)\text{ v }\underset{\sim}{\overline{c}}$$

and suppose that $[0{,}1]$ is divided into three intervals

$$[0, 0, 2]\ .\ [0, 2, 0.3]\ ,\ [0.3, 1]\ .$$

First we consider the interval $[0.\,0, 2]$.

Hypothesis I: $\underset{\sim}{a}\ \wedge\ \underset{\sim}{\overline{b}}\ > \underset{\sim}{\overline{a}}\ \wedge \underset{\sim}{\overline{c}}\ \ .\ \underset{\sim}{a} \wedge\ \underset{\sim}{\overline{b}}\ > \underset{\sim}{\overline{c}}$.

one then has 1) $0 \leq\ \underset{\sim}{a}\ \wedge\ \underset{\sim}{\overline{b}}\ < 0, 2$.

So $0 \leq \text{MIN}\left(\underset{\sim}{a}\,, 1-\ \underset{\sim}{b}\ \right) < 0, 2$.

$\underset{\sim}{a} \geq 0$ and $\underset{\sim}{b}\ \leq\ 1$

and $\underset{\sim}{a} \leq 0, 2$ or / and $\underset{\sim}{b}\ >\ \ 0.8$,

Hypothesis II: $\underset{\sim}{a}\ \wedge\ \underset{\sim}{c} > \underset{\sim}{a}\ \ \wedge \underset{\sim}{b}\ \ \ .\ \underset{\sim}{\overline{a}}\ \wedge\ \underset{\sim}{c} > \underset{\sim}{\overline{c}}$.

One then has $0 \leq\ \ \underset{\sim}{\overline{a}}\ \wedge\ \underset{\sim}{c} < 0.2$,

So $0 \leq \text{MIN}\left(1-\ \underset{\sim}{a}\,, \underset{\sim}{c}\ \right) < 0, 2$.

$\underset{\sim}{a} \leq\ 1$ and $\underset{\sim}{c}\ \geq\ 0$.

and $\underset{\sim}{a} \geq 0, 8$ or / and $\underset{\sim}{c}\ \ \ < 0, 2$.

Hypothesis III : $\underset{\sim}{\overline{c}} > \underset{\sim}{a}\ \ \wedge\ \underset{\sim}{\overline{b}}\ \ .\ \underset{\sim}{\overline{c}}\ \ > \underset{\sim}{\overline{a}}\ \wedge\ \underset{\sim}{c}$.

One then has $0 \leq\ \ \underset{\sim}{\overline{c}} < 0, 2$.

So $0 \leq 1-\ \underset{\sim}{c} < 0, 0\ .\ 2$.

$0.8 < \underset{\sim}{c} \leq 1$.

Now we consider the interval $[0{,}2,\ 0.3]$.

Hypothesis I: $\underset{\sim}{a}\ \wedge\ \underset{\sim}{\overline{b}} > \underset{\sim}{\overline{a}} \wedge \underset{\sim}{c}\ \ .\ \underset{\sim}{a} \wedge\ \underset{\sim}{\overline{b}} > \underset{\sim}{\overline{c}}$.

has $0.2\ \leq\ \underset{\sim}{a}\ \wedge\ \underset{\sim}{\overline{b}} < 0, 3$.

$\underset{\sim}{a} \geq 0.2$ and $\underset{\sim}{b}\ \leq\ \ 0, 8$.

$\underset{\sim}{a} < 0.3$ or / and $\underset{\sim}{b}\ >\ 0.7$.

Hypothesis II: $\underset{\sim}{\overline{a}}\ \ \wedge\ \underset{\sim}{c} > \underset{\sim}{a}\ \ \wedge \underset{\sim}{\overline{b}}.\ \ \underset{\sim}{\overline{a}}\ \wedge\ \underset{\sim}{c} > \underset{\sim}{\overline{c}}$.

$$0.2 \leq \overline{\underset{\sim}{a}} \wedge \underset{\sim}{c} < 0.3,$$

$$\underset{\sim}{a} \leq 0.8 \text{ and } \underset{\sim}{c} \geq 0,2..$$

and $$\underset{\sim}{a} > 0.7 \text{ and } \underset{\sim}{c} < 0,3.$$

Hypothesis III : $\underset{\sim}{c} > \underset{\sim}{a} \wedge \overline{\underset{\sim}{b}} \ . \overline{\underset{\sim}{c}} > \overline{\underset{\sim}{a}} \wedge \underset{\sim}{c}.$

$$0.2 \leq \overline{\underset{\sim}{c}} < 0,3.$$

$$\underset{\sim}{c} \leq 0,8 \text{ and } \underset{\sim}{c} > 0,7.$$

Last;y, we consider the interval $[0,3,1]$.

Hypothesis I: $\underset{\sim}{a} \wedge \overline{\underset{\sim}{b}} > \overline{\underset{\sim}{a}} \wedge \underset{\sim}{c} \ . \underset{\sim}{a} \wedge \overline{\underset{\sim}{b}} > \overline{\underset{\sim}{c}}$

$$0.3 \leq \underset{\sim}{a} \wedge \overline{\underset{\sim}{b}} \leq 1.$$

$$\underset{\sim}{a} \geq 0.3 \text{ and } \underset{\sim}{b} \leq 0,7.$$

$$\underset{\sim}{a} \leq 1 \text{ or / and } \underset{\sim}{b} \geq 0.$$

Hypothesis II: $\overline{\underset{\sim}{a}} \wedge \underset{\sim}{c} > \underset{\sim}{a} \wedge \overline{\underset{\sim}{b}}. \ \overline{\underset{\sim}{a}} \wedge \underset{\sim}{c} > \overline{\underset{\sim}{c}}$.

$$0.3 \leq \overline{\underset{\sim}{a}} \wedge \underset{\sim}{c} < 1,$$

$$\underset{\sim}{a} \leq 0.7 \text{ and } \underset{\sim}{c} \geq 0,3.$$

and $$\underset{\sim}{a} \geq 0 \text{ or / and } \underset{\sim}{c} \leq 1.$$

Hypothesis III: $\overline{\underset{\sim}{c}} > \underset{\sim}{a} \wedge \overline{\underset{\sim}{b}}. \ \underset{\sim}{c} > \overline{\underset{\sim}{a}} \wedge \underset{\sim}{c}$.

$$0.3 \leq \overline{\underset{\sim}{c}} \leq 1.$$

$$\underset{\sim}{c} \leq 0.7 \text{ and } \underset{\sim}{c} \geq 0.$$

Finally, the results of this example may be regrouped in the following fashion:

$$0 \leq \underset{\sim}{f}\left(\underset{\sim}{a} . \underset{\sim}{b} . \underset{\sim}{c}\right) < 0,2.$$

Property: I (1)

$$\left[\left(\underset{\sim}{a} \geq 0\right) \text{and} \left(\underset{\sim}{b} \leq 1\right)\right] \text{or / and} \left[\left(\underset{\sim}{a} \leq 1\right) \text{and} \left(\underset{\sim}{c} \geq 0\right) \text{or / and} \left(\underset{\sim}{c} \leq 1\right)\right].$$

Property: I (2)

$$\left[\left(\underset{\sim}{a} < 0.2\right) \text{or / and} \left(\underset{\sim}{b} > 0.8\right)\right] \text{and} \left[\left(\underset{\sim}{a} > 0.8\right) \text{or / and} \left(\underset{\sim}{c} < 0.2\right) \text{ and } \left(\underset{\sim}{c} > 0.8\right)\right]$$

If properties I (1) (34.64) and I (2) (34.66) are verified, then one has (34.64) (34.67)

$$= 0,2 \leq \underset{\sim}{f}\left(\underset{\sim}{a} . \underset{\sim}{b} . \underset{\sim}{c}\right) < 0,3.$$

Property: I (2)

$\left[\left(\underset{\sim}{a} \geq 0,2\right) \text{and} \left(\underset{\sim}{b} \leq 0,8\right)\right]or / and\left[\left(\underset{\sim}{a} \leq 0.8\right) \text{and} \left(\underset{\sim}{c} \geq 0.2\right) \text{or/and} \left(\underset{\sim}{c} \leq 0.8\right)\right]$.

$\left[\left(\underset{\sim}{a} < 0.3\right) \text{or / and} \left(\underset{\sim}{b} > 0.7\right)\right]$ and $\left[\left(\underset{\sim}{a} > 0.7\right) \text{or/and} \left(\underset{\sim}{c} < 0.3\right) \text{ and } \left(\underset{\sim}{c} > 0.7\right)\right]$.

If properties $I_1^{(2)}$ and $I_2^{(2)}$ are satisfied, then one has $0,2 \leq \underset{\sim}{f}\left(\underset{\sim}{a}\,.\underset{\sim}{b}\,.\underset{\sim}{c}\right) < 0,3$.

$$0,3 \leq \underset{\sim}{f}\left(\underset{\sim}{a}\,.\underset{\sim}{b}\,.\underset{\sim}{c}\right) \leq 1 .$$

Property: $I_1^{(3)}$

$\left[\left(\underset{\sim}{a} \geq 0,3\right) \text{and} \left(\underset{\sim}{b} \leq 0,7\right)\right]or / and\left[\left(\underset{\sim}{a} \leq 0.7\right) \text{and} \left(\underset{\sim}{c} \geq 0.3\right) \text{or/and} \left(\underset{\sim}{c} \leq 0.7\right)\right]$.

Property: $I_2^{(2)}$

$\left[\left(\underset{\sim}{a} \leq 1\right) \text{or / and} \left(\underset{\sim}{b} > 0\right)\right]$ and $\left[\left(\underset{\sim}{a} \geq 0\right) \text{or/and} \left(\underset{\sim}{c} \leq 1\right) \text{ and } \left(\underset{\sim}{c} \geq 0\right)\right]$.

If properties $I_1^{(3)}$ and $I_2^{(3)}$are satisfied, then one has (34.70).

Important Remark.

We examine I_1 (34.22) and I_1 (34.23). One may see that properties I_1 and I_2 are dual to one another if one mutually replaces.

(<) by (≥), (≤)by (>) , (>) by (≤) , (≥) by (<)

(and) by (and / or). (and / (or) by (and)

(for the last two, $>$ becomes $\geq$ and $<$ becomes $\leq$. since the last interval is closed both on the left and the right).

This property is not fortuitous; it is general for all reduced polynomial forms with respect to v or with respect to ∧ .

$$\underset{\sim}{f}\,(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) = (\underset{\sim}{a} \text{ v } \underset{\sim}{b}) \wedge (\overline{\underset{\sim}{b}} \text{ v } \underset{\sim}{c}).$$

under what conditions does one have

$$\alpha_{k-1} < \underset{\sim}{f}\,(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) < \alpha_k$$

Hypothesis 1: $\underset{\sim}{a} \text{ v } \underset{\sim}{b} < \overline{\underset{\sim}{b}} \text{ v } \underset{\sim}{c}$

This implies

$$\alpha_{k-1} \leq \underset{\sim}{a} \text{ v } \underset{\sim}{b} < \alpha_k .$$

$$\alpha_{k-1} \leq MAX\,(\underset{\sim}{a} \quad \underset{\sim}{b}) < \alpha_k .$$

Since we may not arbitrarily place $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$ with respect to one another, it is necessary that

$\underset{\sim}{a} \geq \alpha_{k-1}$ or / and $\underset{\sim}{b} \geq \alpha_{k-1}$.

and $\underset{\sim}{a} < \alpha_k$ and $\underset{\sim}{b} < \alpha_k$.

Hypothesis II: $\underset{\sim}{\bar{b}}$ v $\underset{\sim}{c} > \underset{\sim}{a}$ v $\underset{\sim}{b}$.

This implies

$$\alpha_{k-1} \leq \underset{\sim}{\bar{b}} \text{ v } \underset{\sim}{c} < \alpha_k .$$

or again

$$\alpha_{k-1} \leq MAX\ (1 - \underset{\sim}{b}\ ,\ \underset{\sim}{c}) < \alpha_k .$$

Thus $\underset{\sim}{b} \leq 1\alpha_{k-1}$ or / and $\underset{\sim}{c} \geq \alpha_{k-1}$.

and $\underset{\sim}{b} > 1 - \alpha_k$ and $\underset{\sim}{c} < \alpha_k$.

Regrouping the results obtained, we have:

Property: $I_1^{(1)}$

$$\left[\left(\underset{\sim}{a} \geq \alpha_{k-1}\right) \text{or / and} \left(\underset{\sim}{b} \geq \alpha_{k-1}\right)\right] \text{and} \left[\left(\underset{\sim}{b} \leq 1 - \alpha_{k-1}\right) \text{or / and} \left(\underset{\sim}{c} \geq \alpha_{k-1}\right)\right] .$$

Property: $I_2^{(1)}$

$$\left[\left(\underset{\sim}{a} < \alpha_k\right) \text{and} \left(\underset{\sim}{b} < \alpha_k\right)\right] \text{or / and} \left[\left(\underset{\sim}{b} > 1 - \alpha_k\right) \text{and} \left(\underset{\sim}{c} < \alpha_k\right)\right] .$$

In order that $\alpha_{k-1} < \underset{\sim}{f}\ (\underset{\sim}{a}\, ,\underset{\sim}{b}\, ,\underset{\sim}{c}) < \alpha_k$

be satisfied. It is necessary and sufficient that properties I; and II be satisfied.

We not that the property of duality reappears, but *or / and* has taken the place of *and*, and vice versa.

34. LOGICAL STRUCTURE OF A FUNCTION OF FUZZY VARIABLES

Recall that the propositional algebra, in which appear the propositions

"and" denoted by Δ

"or / and"denoted by ∇

"complement" denoted by $-$

follows exactly the same rules as those of Boolean algebras.

Δ is associated with $\cap$

∇ is associated with $\cup$

$= -$ is associated with $-$.

In order to present the logical structure of the relations (strict) or nonstrict inequalities) that appear in a fuzzy logical function, and considering an interval

$[(\alpha_{k-1}, \alpha_k)]$ we will use the following symbols.

Let $\underset{\sim}{f}\left(\underset{\sim}{a}, \underset{\sim}{b}, \dots \dots \underset{\sim}{l}\right)$ may be presented in a reduced polynomial form with respect to v . In order to obtain the logical structure in the interval (α_{k-1}, α_k) one proceeds as follows:

TABLE OF PRINCIPAL FUNCTION OF TWO FUZZY VARIABLES AND OF THEIR LOGICAL STRUCTURES FOR AN INTERVAL $[a_{k-1}, a_k]$.

$\underset{\sim}{f}(\underset{\sim}{a}, \underset{\sim}{b})$	Polynomial form with respect to ∇	Polynomial form with respect to ∇
$a \wedge b$	$(\mathcal{R}_a \Delta \mathcal{R}'_a \Delta \mathcal{R}_b) \nabla (\mathcal{R}_a \Delta \mathcal{R}_b \Delta \mathcal{R}'_b)$	$(\mathcal{R}_a) \Delta (\mathcal{R}_b) \Delta (\mathcal{R}'_a \nabla \mathcal{R}'_b)$
$\bar{a} \wedge b$	$(\mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_b) \nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}_b \Delta \mathcal{R}'_b)$	$(\mathcal{R}_{\bar{a}}) \Delta (\mathcal{R}_b) \Delta (\mathcal{R}'_{\bar{a}} \nabla \mathcal{R}'_b)$
$\bar{a} \wedge \bar{b}$	$(\mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_{\bar{b}}) \nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_{\bar{b}})$	$(\mathcal{R}_{\bar{a}}) \Delta (\mathcal{R}_{\bar{b}}) \Delta (\mathcal{R}'_{\bar{a}} \nabla \mathcal{R}'_{\bar{b}})$
$a \vee b$	$(\mathcal{R}_a \Delta \mathcal{R}'_a \Delta \mathcal{R}'_b) \nabla (\mathcal{R}'_a \Delta \mathcal{R}_b \Delta \mathcal{R}'_b)$	$(\mathcal{R}'_a) \Delta (\mathcal{R}'_b) \Delta (\mathcal{R}_a \nabla \mathcal{R}_b)$
$\bar{a} \vee b$	$(\mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}'_b) \nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}_b \Delta \mathcal{R}'_b)$	$(\mathcal{R}'_{\bar{a}}) \Delta (\mathcal{R}'_b) \Delta (\mathcal{R}_{\bar{a}} \nabla \mathcal{R}_b)$
$\bar{a} \vee \bar{b}$	$(\mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}'_{\bar{b}}) \nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_{\bar{b}})$	$(\mathcal{R}'_{\bar{a}}) \Delta (\mathcal{R}'_{\bar{b}}) \Delta (\mathcal{R}_{\bar{a}} \nabla \mathcal{R}_{\bar{b}})$
$(a \wedge b) \vee (\bar{a} \wedge b)$	$(\mathcal{R}_a \Delta \mathcal{R}'_a \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_{\bar{b}}) \nabla (\mathcal{R}_a \Delta \mathcal{R}'_a \Delta \mathcal{R}'_b \Delta \mathcal{R}_{\bar{b}})$ $\nabla (\mathcal{R}_a \Delta \mathcal{R}'_b \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_{\bar{b}}) \nabla (\mathcal{R}_a \Delta \mathcal{R}'_b \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_{\bar{b}})$ $\nabla (\mathcal{R}'_a \Delta \mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_b) \nabla (\mathcal{R}'_a \Delta \mathcal{R}_{\bar{a}} \Delta \mathcal{R}_b \Delta \mathcal{R}'_b)$ $\nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_b \Delta \mathcal{R}'_{\bar{b}}) \nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}_b \Delta \mathcal{R}'_b \Delta \mathcal{R}'_{\bar{b}})$	$(\mathcal{R}_a \nabla \mathcal{R}_{\bar{a}}) \Delta (\mathcal{R}_a \nabla \mathcal{R}_b) \Delta (\mathcal{R}_{\bar{a}} \nabla \mathcal{R}_{\bar{b}})$ $\Delta (\mathcal{R}_b \nabla \mathcal{R}_{\bar{b}}) \Delta (\mathcal{R}'_a \nabla \mathcal{R}'_{\bar{b}})$ $\Delta (\mathcal{R}'_{\bar{a}} \nabla \mathcal{R}'_b)$
$(a \vee \bar{b}) \wedge (\bar{a} \vee b)$	$(\mathcal{R}_a \Delta \mathcal{R}'_a \Delta \mathcal{R}_b \Delta \mathcal{R}'_{\bar{b}}) \nabla (\mathcal{R}_a \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_b \Delta \mathcal{R}'_b)$ $\Delta (\mathcal{R}_a \Delta \mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_a \Delta \mathcal{R}'_{\bar{b}}) \nabla (\mathcal{R}_a \Delta \mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}'_b)$ $\Delta (\mathcal{R}'_a \Delta \mathcal{R}_b \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_b) \nabla (\mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_b \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_b)$ $\nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_a \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_b) \nabla (\mathcal{R}_{\bar{a}} \Delta \mathcal{R}'_{\bar{a}} \Delta \mathcal{R}_{\bar{b}} \Delta \mathcal{R}'_b)$	$(\mathcal{R}_a \nabla \mathcal{R}_{\bar{b}}) \Delta (\mathcal{R}_b \nabla \mathcal{R}_{\bar{a}}) \Delta (\mathcal{R}'_a \nabla \mathcal{R}'_b)$ $\Delta (\mathcal{R}'_a \nabla \mathcal{R}'_{\bar{a}}) \Delta (\mathcal{R}'_b \nabla \mathcal{R}'_{\bar{b}}) \Delta (\mathcal{R}'_{\bar{a}} \nabla \mathcal{R}'_b)$.

38. FUZZY PROPOSITIONS AND THEIR FUNCTIONAL REPRESENTATION

Fuzzy logic does not rest on truth tables as does formal logic, but upon operations realized on fuzzy subsets.

We begin with a comparative example based on the tale, "Little Red Riding Hood." † We comider two formal propositions for which one must verify a posteriori (after reading the story) whether they are true or false:

$\wp_1$: the wolf is dressed in the guise of a grandmother

$\wp_2$: the wolf has eaten the little girl.

The proposition $\wp_1 \Delta \wp_2$ will mean∗": "the wolf is dressed as a grandmother and has eaten the little girl. In order that this be true, it is proper that the two statements be true; if only one or neither is true, this would not be coherent with the story of Little Red Riding Hood. Thus we have the truth table

†Pardon this rather naive example, which constitutes a very elementary didactic explication. Fuzzy logic will be considered as an application in the following volume.

∗On the subject of the use of the symbols Δ and ∇, we refer to the fondnote on p. 214 at the beginning of Section 35.

$\wp_1$	$\wp_2$	$\wp_1 \Delta \wp_2$
true	true	true
true	false	false
false	true	false
false	false	false

Fig. 38.1

But we now present the two logical statements in another fashion. There exists a set of animals

$$E = \{cat, dog, wolf, fox, goat, rat, rabbit\},$$

Consider $A \subset E$, the formal subset of animals apt to dress as a grandmother:

$$A = \{(cat\ |0{,}1), (dog\ |\ 1), (wolf\ |1), (fox\ |0{,}5), (goat\ |1), (rat\ |0), (rabbit\ |0)\}$$

that is,

$$A = \{wolf\}$$

Consider $B \subset E$ the formal subset of animals likely to eat the little girl:

$$B = \{(cat\,|0), (dog\,|0), (wolf\,|1), (fox\,|0), (goat\,|0), (rat\,|0), (rabbit\,|0)\}.$$

that is,

$$B \cap \{wolf\}.$$

The formal subset of animals prone to dressing as a grandmother and eating little girls is

$$A \cap B = \{wolf\}.$$

Thus, through such a procedure we have verified that the wolf is indeed the cunning and cruel animal described in this celebrated story.

We consider now two statements from the fuzzy tale of Little Red Riding Hood. There exists a set of animals:

$$E = \{cat, dog, wolf, fox, goat, rat, rabbit\}$$

Consider $A \subset E$, the fuzzy subset of animals apt to dress as a grandmother:

$$A = \{(cat\,|0{,}1), (dog\,|\,1), (wolf\,|1), (fox\,|0{,}5), (goat\,|\,1), (rat\,|0), (rabbit\,|0)\}.$$

Consider $B \subset E$, the fuzzy subset of animals likely to eat a little girl:

$$B = \{(cat\,|0{,}1), (dog\,|0{,}4), (wolf\,|\,1), (fox\,|0{,}7), (goat\,|0), (rat\,|0), (rabbit\,|\,0)\}.$$

Then, the fuzzy subset of animals apt to dress as a grandmother and eat a little girl will be

$$A \cap B = \{(cat\,|0{,}1), (dog\,|0{,}4), (wolf\,|\,1), (fox\,|0{,}5), (goat\,|0), (rat\,|0), (rabbit\,|0)\}.$$

The tale may refer to the wolf, but also to a fox, a dog, or a cat.

The statements of the fuzzy logic, as the statements of the formal logic, are associated explicitly or implicitly to set theory, fuzzy for the former and formal for the latter.

With the operations, $\cap$,$\cup$,, and$-$ (intersection, union, and complementation) one associates in the formal logic the connectives Δ, ∇ and$\neg$ (conjunction and, disjunction or and, negation not).

Passage to the fuzzy, connectives Δ, ∇ and$\neg$of the corresponding fuzzy logic does not present any difficulties since we have already defined the corresponding set operations in Section 5.

But it is necessary to give special attention to the other connectives:

implication

metaimplication

logical equivalence

We now go on to review these questions, first in formal logic then in fuzzy logic.

Consider two formal propositions $\wp$ and $\succsim$. The compound proposition "$\wp$ implies $\succsim$." Denoted $\wp > \succsim$, corresponds to the truth table in Figure 39.2.

$\mathfrak{P}$	$\mathfrak{Q}$	$\mathfrak{P} > \mathfrak{Q}$
false	false	true
false	true	true
true	false	false
true	true	true

Fig. 38.2

To this compound proposition corresponds, for the subset A associated with $\wp$ and the subset B associated with $\succsim$, the set operation $\bar{A} \cup B$.

Now we consider the compound proposition "$\wp$ metaimplies $\succsim$," denoted $\wp \Rightarrow \succsim$. To this metaimplication one gives the following sense: when $\wp$ is true, $\succsim$ is always true (the syllogism rule is happily recovered here), but one may affirm nothing when $\wp$ is false, $\succsim$ may be just as well be true as false, Thus, a statement like "if the sea is made of sweet cider, I will change myself into a siren" is correct, the sea being, alas, evil to drink and certainly not made of sweet cider. The connection $\Rightarrow$ therefore reduces to: if $\wp \Rightarrow \succsim$ is necessary that $\wp$ be true only when $\succsim$ is also.

One must guard therefore against confusing $\wp > \succsim$ and $\wp \Rightarrow \succsim$. The first is an operation of logic.

$$\wp > \succsim = \bar{\wp} \nabla \succsim \ (in\ one\ notion)$$

$$= (\neg \wp) \nabla (\succsim) (in\ another\ notion)$$

The second is a metalogical operation that may not be brought to (39.11). But the habit has been taken of calling metaimplication implication and thus confusing the two. The compound proposition$\wp > \succsim$ does not introduce a cause and effect relation, nor a proof of 2 with respect to 2, contrary to that which holds for $\wp \Rightarrow \succsim$

One may present the false paradox introduced by $\wp > \succsim$ in the following terms: since the propositions $\wp$and$\succsim$ have not been analyzed, since they occur only through their contents, since the only given accessible is the logical value of each, $\wp > \succsim$ may not introduce a relation of cause and effect. But if one knows a priori that$\wp$ is true and that $\wp > \succsim$ is true, then one may conclude that$\succsim$ is true.

We present an example cited in reference [3K].Let$\wp$ and$\succsim$ be the following propositions, which we shall examine considering the table of Figure 39.2:

$\wp$: Napoléon died at Saint-Hélène(true)

$\succsim$: Vercingetorix wore a moustache (one is not sure)

$\wp > \succsim$: true if $\succsim$ is true

$\wp$: Two and two are five (false)

$\succsim$: 2 is a prime number (false)

$\wp > \succsim$: is true

$\wp$: The moon is made of gruyère cheese(false)

$\succsim$: 17 is prime(true)

$\wp > \succsim$: is true

$\wp$: 17 is prime(true)

$\succsim$: 16 is prime(false)

$\wp > \succsim$: is false

Logical equivalence is less ambiguous. This will be defined by the truth table in Figure 38.3.

$\mathfrak{P}$	$\mathfrak{Q}$	$\mathfrak{P} \equiv \mathfrak{Q}$
false	false	true
false	true	false
true	false	false
true	true	true

Fig. 38.3

Like implication, logical equivalence does not bring the contents of the two propositions into a causal relationship.

To this compound proposition corresponds, for the subset A associated with$\wp$ and the subset B associated with $\gtrsim$, the set operation $(\bar{A} \cup B) \cap \ (A \cup \bar{B})$.

Metaequivalence carries the same name usually:$\wp$ is equivalent to $\gtrsim$, denoted$\wp \Rightarrow\gtrsim$ that is,$\wp$ metaimplies$\gtrsim$ and $\gtrsim$ metaimplies$\wp$, because the symmetry leads to a truth table identical to that of equivalence$\wp \equiv\gtrsim$. This is why one may confuse these without ambiguity.

Fuzzy propositions of the types fuzzy implication and fuzzy equivalence will be defined respectively with reference to the operations $\underline{\overline{A}} \cup \underline{B}$and $\left(\underline{A} \cup \underline{\overline{B}}\right) \cap \left(\underline{\overline{A}} \cup \underline{B}\right)$, We insist on the fact that one mast, as for intersection, union, and negation, pass through the reference set and the associated membership set.

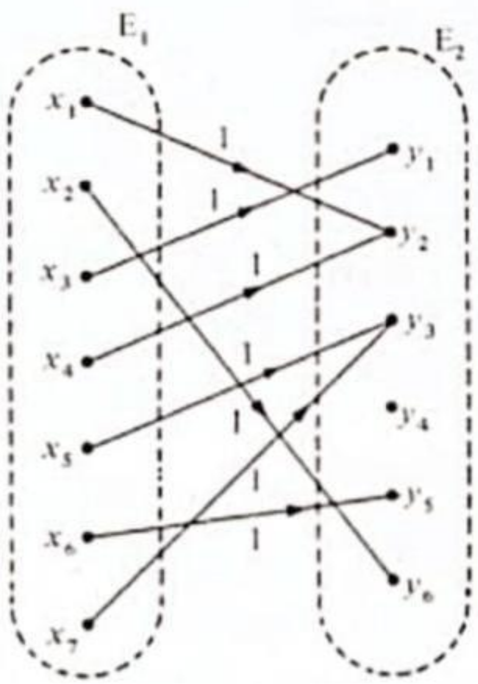
In order to define a metaimplication in furzy logic we shall use the notion of a binary relation. A as that represented in Figures 39.4 and 39.5 gives an example where $x_i \in E_1, y_i \in E_2$. One sees evident here:

$$if\ x = x_1\ then\ y = y_2,$$

$$if\ x = x_2\ then\ y = y_6,$$

$$if\ x = x_3\ then\ y = y_1,$$

$$if\ x = x_7\ then\ y = y_1,$$



$E_1 \backslash E_2$	y_1	y_2	y_3	y_4	y_5	y_6
x_1	0	1	0	0	0	0
x_2	0	0	0	0	0	1
x_3	1	0	0	0	0	0
x_4	0	1	0	0	0	0
x_5	0	0	1	0	0	0
x_6	0	0	0	0	1	0
x_7	0	0	1	0	0	0

Fig. 38.4 Fig. 38.5

Figure 38.6, on the other hand, will correspond to an element of E_1 a fuzzy subset of E_2:

If $x = x_1$ then $\underline{B} = \{(y_1|0{,}8), (y_2|1), (y_3|0{,}3), (y_4|1), (y_5|0{,}9), (y_6|0{,}9)\}$,

If $x = x_2$ then $\underline{B} = \{(y_1|0{,}2), (y_2|0{,}9), (y_3|1), (y_4|0), (y_5|0{,}6), (y_6|1)\}$,

If $x = x_3$ then $\underline{B} = \{(y_1|0{,}3), (y_2|0{,}8), (y_3|0{,}9), (y_4|1), (y_5|0{,}8), (y_6|0{,}1)\}$,

… … …

If $x = x_7$ then $\underline{B} = \{(y_1|0{,}1), (y_2|1), (y_3|0), (y_4|0{,}9), (y_5|0{,}3), (y_6|1)\}$

But in Section 15 we have defined the possibility of a correspondence between fuzzy subsets where $\underline{A} \subset E_1$and $\underline{B} \subset E_2$; this was done with the aid of the notion of a conditioned fuzzy subset The relation giving the fuzzy subset $\underline{B}$ corresponding to the fuzzy subset $\underline{A}$ is then

$E_1 \backslash E_2$	y_1	y_2	y_3	y_4	y_5	y_6
x_1	0,8	1	0,3	1	0,9	0,9
x_2	0,2	0,9	1	0	0,6	1
x_3	0,3	0,8	0,9	1	0,8	0
x_4	0,5	0	1	1	0,8	0,9
x_5	1	0,2	0,9	0,6	0	0,5
x_6	0,6	0,8	1	1	0,8	1
x_7	0,1	1	0	0,9	0,3	1

Fig. 38.6

$$\left(\mu_{\underline{B}}(y)\right) = \operatorname*{MAX}_{x \in E_1} \text{MIN}\left(\mu_{\underline{B}}(y|x).\mu_{\underline{A}}(x)\right)$$

We have given an example in Section 15 [see (15.3)-(15.11)]; here we take up another example using the fuzzy relation of Figure 38.6.

Suppose that

$$\underline{A} = \{(x_1|0{,}2),(x_2|0{,}3),(x_3|0{,}5),(x_4|1),(x_5|0),(x_6|0),(x_7|0{,}8)\}$$

One sees successively

$$\mu_{\underline{B}}(y_1) = \text{MAX}[MIN\,(0{,}8\,.\;0{,}2), MIN(0{,}2\,.\,0{,}3), MIN(0{,}3\,.\;0{,}5), MIN(0{,}5\,.\;1),$$
$$MIN(1\,.\;0), MIN(0{,}6\,.\;0), MIN(0{,}1\,.\;0{,}8)]\,.$$

$$= MAX[0{,}2\,;0{,}2;0{,}5;0;0;0{,}1] = 0{,}5.$$

$$\mu_{\underline{B}}(y_2) = \text{MAX}[MIN\,(1\,.\;0{,}2), MIN(0{,}9\,.\,0{,}3), MIN(0{,}8\,.\;0{,}5), MIN(0,\;\;1),$$
$$MIN(0{,}2\,.\;0), MIN(0{,}8\,.\;0), MIN(1\,.\;0{,}8)]\,.$$

$$= MAX[0{,}2\,;0{,}3;0{,}5;0;0;0{,}0{,}8] = 0{,}8.$$

and in the same manner

$$\mu_{\underline{B}}(y_3) = 1, \mu_{\underline{B}}(y_4) = 1, \mu_{\underline{B}}(y_5) = 0{,}8\ , \mu_{\underline{B}}(y_6) = 0{,}9$$

The calculations have been presented in Figure 39.7, where the operation * corresponds to max-min.

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ \hline 0{,}2 & 0{,}3 & 0{,}5 & 1 & 0 & 0 & 0{,}8 \end{array} \ * \ \begin{array}{c|cccccc} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ \hline x_1 & 0{,}8 & 1 & 0{,}3 & 1 & 0{,}9 & 0{,}9 \\ x_2 & 0{,}2 & 0{,}9 & 1 & 0 & 0{,}6 & 1 \\ x_3 & 0{,}3 & 0{,}8 & 0{,}9 & 1 & 0{,}8 & 0 \\ x_4 & 0{,}5 & 0 & 1 & 1 & 0{,}8 & 0{,}9 \\ x_5 & 1 & 0{,}2 & 0{,}9 & 0{,}6 & 0 & 0{,}5 \\ x_6 & 0{,}6 & 0{,}8 & 1 & 1 & 0{,}8 & 1 \\ x_7 & 0{,}1 & 1 & 0 & 0{,}9 & 0{,}3 & 1 \end{array} \ = \ \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ \hline 0{,}5 & 0{,}8 & 1 & 1 & 0{,}8 & 0{,}9 \end{array}$$

Fig. 38.7

One therefore has: if

$$\underline{A} = \{(x_1|0{,}2), (x_2|0{,}3), (x_3|0{,}5), (x_4|1), (x_5|0), (x_6|0), (x_7|0{,}8)\}$$

then

$$\underline{B} = \{(y_1|0{,}5), (y_2|0{,}8), (y_3|1), (y_4|1), (y_5|0{,}8), (y_6|0{,}9)\}$$

In this fashion we show that considering an if-then proposition corresponds well to what is used in formal relations.

Let

$$A = \{(x_1|0), (x_2|0), (x_3|0), (x_4|1), (x_5|0), (x_6|0), (x_7|0)\}$$

that is

$$A = \{x_4\}$$

Referring to the correspondence again, one finds

$$B = \{(y_1|0), (y_2|1), (y_3|0), (y_4|0), (y_5|0), (y_6|0)\}$$

that is,

$$B = \{y_2\}$$

which may be stated

$$if\ A = \{x_4\}, then\ B = \{y_2\}$$

or again

$$if\ x = x_4, then\ y = y_2$$

We review all the propositions stated thus far:

fuzzy conjunction (fuzzy and): defined by $\underline{A} \cap \underline{B}$.

fuzzy disjunction (fuzzy or): defined by $\underline{A} \cup \underline{B}$.

fuzzy negation (fuzzy not): defined by $\overline{\underline{A}}$.

fuzzy implication: defined by$\overline{\underline{A}} \cup \underline{B}$.

fuzzy equivalence: defined by $\left(\underline{A} \cup \overline{\underline{B}}\right) \cap \left(\overline{\underline{A}} \cup \underline{B}\right)$.

fuzzy if-then: defined by $\mu_{\underline{B}}(y) = \underset{x}{\mathrm{MAX}}\, MIN\ \left(\mu_{\underline{B}}(y|x).\mu_{\underline{A}}(x)\right)$

(fuzzy metaimplication)

This last was not a fuzzy logic, but rather a fuzzy metalogic, proposition.

In Volume II, devoted to applications of the theory of fuzzy subsets, various sections will reconsider these notions in detail and will give a number of developments.

39. THE THEORY OF FUZZY SUBSETS AND THE THEORY OF PROBABILITY

Many persons, without too much thought, state: Why be interested in the theory of fuzzy subsets? The theory of probability serves very well for all that. There are, in fact, several common aspects between the two theories; but these theories relate to some considerations that it is appropriate to distinguish. We proceed first to review the basics of the theory of probability and then examine that which joins and that which separates these theories.

Axiomatics of the theory of probability.

(1) Case of a finite reference set. Let E be a finite reference set, I. (E) its finite power set, and Δ a subset of I (E) necessarily containing E. The subset Δ will be called a family and one will say that this family is probabilizable if the following two conditions are satisfied:

a) $\forall \wedge \in \Delta : \ \overline{\mathrm{A}} \in \Delta$.

b) $\forall \wedge \in \Delta$ and $\cup B \in \Delta$.

For example, let

$\mathrm{E} = \{a, b, c, d\}$.

$\Delta = \{(\emptyset. (\mathrm{b}). (\mathrm{c}). (\mathrm{b.c})(\mathrm{a.d}). (\mathrm{a.b.d}). (\mathrm{a.c.d}). \mathrm{E}\}$.

The family Δ is probabilizable.

Theory of fuzzy subsets and theory of probability

Properties (40.1) and (40.2) imply several others as the reader may easil prove.

c) $\emptyset \in \Delta$.

d) $\forall$ A and $\forall$ B : $\mathrm{A} \cap \mathrm{B} \in \Delta$.

c) $\mathrm{A} - \mathrm{B} = \mathrm{A} \cap \overline{\mathrm{B}} \in \Delta$.

A probabilizable family Δ constitutes a ring for the operations. (disjunctive sim) and $\cap$ (intersection). Indeed, one verifies:

$\forall$ A . B.C $\in \Delta$

$(\mathrm{A} * \mathrm{B}) * \mathrm{C} = \mathrm{A} * (\mathrm{B} * \mathrm{C})$ associativity for $*$;

$\mathrm{A} * \emptyset = \emptyset * \mathrm{A} = \mathrm{A}$. $\emptyset$ is the identity for $*$;

$\mathrm{A} * \mathrm{A} = \emptyset$, every A has a inverse (it its own inverse);

$\mathrm{A} * \mathrm{B} = \mathrm{B} * \mathrm{A}$. commutativity.

Thus we have a commutative group with respect to the operation $*$. On the other hand,

$(\mathrm{A} \cap \mathrm{B}) \cap \mathrm{C} = \mathrm{A} \cap (\mathrm{B} \cap \mathrm{C})$. associativity for $\cap$;

And finally

$(\mathrm{A} * \mathrm{B}) \cap \mathrm{C} = (\mathrm{A} \cap \mathrm{C}) * (\mathrm{B} \cap \mathrm{C})$. distributivity on the left and right

$C \cap (A * B) = (C * A) * (C \cap B)$. with respect to $*$.

Thus $(\Delta, *, \cap)$ make up a ring structure.

Finally, any family Δ forms a distributive and complemented lattice, that is, a Boolean lattice, in which the order relation is inclusion. Thus for the family Δ given by (40.4), one obtains the boolean lattice represented in figure 40.1

A subset F (I (E) is called a probability base over E if with respect to F, using complementation and union, one may arrive at a probabilistic family Δ C I (E). One also says that F is a generator of Δ; such a generator is not in general unique.

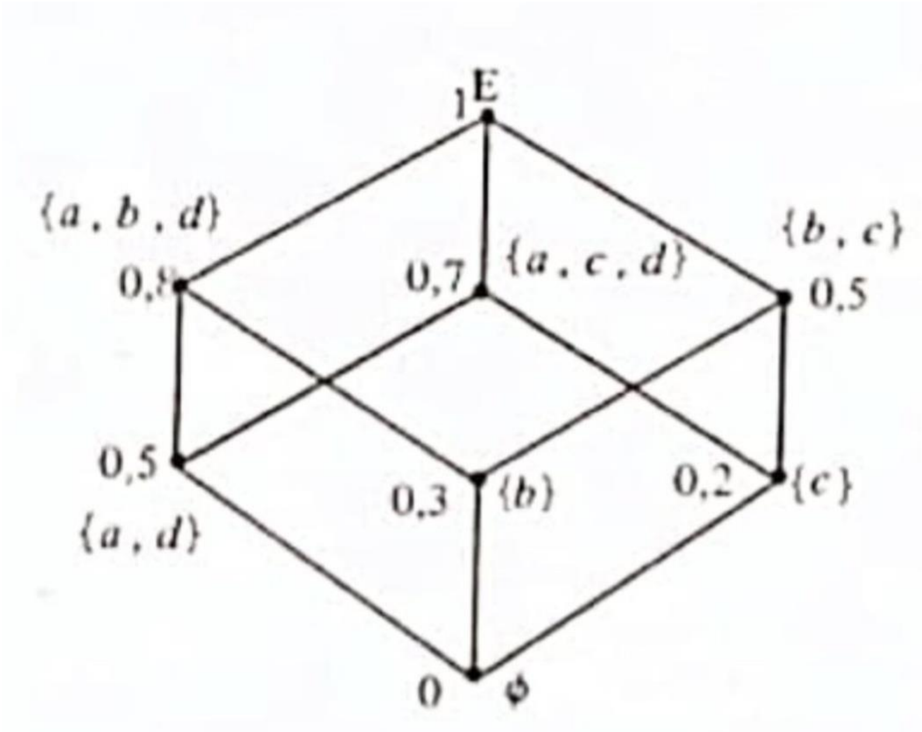


Figure 39.1

For example, referring Figure 39.1. one easily sees that

$F = \{(a, d), (b), (c)\}$.

is a generator of (40.4).

(2) Case of an infinite reference ser (denumerable or not), In this case I (E) is not denumerable; let Δ be a subset of I (E) necessarily containing E. One will say that the family Δ is probabilizable if:

g) $\forall \Lambda \in \Delta : \overline{\mathrm{A}} \in \Delta$.

h) for any denumerable sequence, $A_1 \;..\; A_2 \;\ldots\ldots..\; A_n \;\ldots\ldots$

$A_1 \;..\; A_2 \;\ldots\ldots..\; A_n \;\ldots\ldots \in \Delta \Longrightarrow A_1 \cup A_2 \cup \ldots\ldots.. \cup A_n \cup \ldots\ldots \in \Delta.$

Probability. Theoretical definition. Given a probabilizable family $\Delta \subset$ I (E). a *probability* is a mapping of Δ into R^* having the following properties where the value taken by X in R' is written *pr(x)*:

i) $\forall$ A $\in \Delta$: *pr(A)* ≥ 0.
j) $\forall$ A $\in \Delta$ and $\forall$ B $\in \Delta$: $A \cap B = \emptyset \Longrightarrow$ *pr*(A $\cup$B) = *pr*(A) + *pr*(B).
k) *pr*(E) = 1.

With respect to the five axioms (a), (b), (i), (j), and (k) it is easy to prove a certain number of properties:

$pr(\emptyset)$= 0
pr(A)= 1 + pr(A).
$B \subset A \Longrightarrow\; pr$(B) $\leq pr$(A).

Returning to the nation of a fuzzy subset, we insist on the following important point: "it is not sufficient to associate with a subset a number $p \in [0, 1]$ such hat p is

brobability; it is necessary that the subset and *p* satisfy the five fundamental axioms

mentioned above".

Difference between the probability concept for fuzzy subset, and for ordinary subsets. We consider a very simple finite example. How does one proceed in the theory of fuzzy subsets?

E = $\{a, b, c, d\}$.

One defines a fuzzy subset by assigning to each element a value of the membership function; for example.

A = $\{(a\ 0.3), (b\ 0.7), (c\ 10), (d\ 1)\}$

In probability theory one assigns the numbers $p \in [0, 1]$ to the ordinary subsets constituting a probabilizable family. Thus, letting Δ be given by (40.4) one might have, for example.

$Pr(\emptyset)$ = 0.

$pr\{(b)\}$=0.3. $pr\{(c)\}$=0.2. $pr\{(a,d)\}$=0.5. $pr\{(a,b,d)\}$=0.8.

$pr\{(a,c,d)\}$=0.7. $pr\{(b,c)\}$=0.5. $pr\{$(E)$\}$=1.

As one sees here the two considerations are quite distinct, and one may conceive (and this is useful) of the assignment of probabilities to fuzzy subsets by taking each fuzzy subset belonging to a reference set formed by elements that are fuzzy subsets of another reference set. For example, assign a probability to A and write.

$$Pr(\text{A}) = 0.\ 6$$

One may imagine a probability theory of fuzzy events. But, one must evidently distinguish between the two theories, that of fuzzy subsets and that of the probabilization of ordinary subsets.

The theory of fuzzy events. But, one must evidently distinguish between the two theories, that of fuzzy subsets and that of the probabilization of ordinary subsets.

The theory of fuzzy subsets is related to the theory of a vector lattice, and probability theory to the theory of Boolean lattice.

UNIT V

THE LAWS OF FUZZY COMPOSITION

40.THE LAWS OF FUZZY COMPOSITION

In this chapter, the reader should now go on to consider, is a first introduction to some important developments that will become more and more involving for them.

LAW OF INTERNAL COMPOSITION

The law of internal composition on a set E, is a mapping E x E into E. In other words, to each ordered pair (x,y) $\in E \times E$, one corresponds one and only one element $z \in E$.

LAW OF EXTERNAL COMPOSITION

Let $x \in E_1, y \in E_2$ and $z \in E_3$. A mapping $E_1 \times E_2$ into E_3 is called a law of external composition. In other words, to each ordered pair (x, y) one corresponds an element $z \in E_3$, and only one such element.

If and only if $E_1 = E_2 = E_3$ is the law of composition internal.

Examples :

(i) If $E_1 = E_2 = R$ (the set of real numbers), if the law is that of ordinary addition +, this law is internal since the addition of a real with a real always gives a real; indeed one has $E_3 = R$.

(ii) If $E_1 = E_2$=the set of free vectors in a plane and if one there defines $\times$ as the vector product (cross product) of the two vectors , one has a law of external composition.

GROUPOID

An ordered pair formed by a set E and an internal law of composition * defined on this set is called a groupoid. This is denoted by (E,*).

Examples:

(i) The law of composition presented in the following figure gives a groupoid.

E \ E	A	B	C	D	E
A	B	A	D	D	C
B	C	B	B	A	E
C	A	A	A	B	C
D	C	A	B	B	C
E	E	C	A	A	D

(Fig. 44.1)

(ii) The greatest common divisor and least common multiple of positive integers define internal laws defined throughout the set N_0 of positive integers. If $*_1$ indicates the greatest common divisor and $*_2$ the least common multiple, then $(N_0,*_1)$ and $(N_0,*_2)$ are groupoids.

LAW OF FUZZY INTERNAL COMPOSITION, FUZZY GROUPOID

Let E be a reference set and $\underset{\sim}{A} \subset E$. A law of internal composition on $\underset{\sim}{P}(E)$, that is, a mapping from $\underset{\sim}{P}(E) \times \underset{\sim}{P}(E)$ into $\underset{\sim}{P}(E)$. In other words, to each ordered pair $(\underset{\sim}{A}, \underset{\sim}{B})$, where

$\underset{\sim}{A} \subset E, \underset{\sim}{B} \subset E$ one corresponds a fuzzy subset $\underset{\sim}{C} \subset E$ and only one. If m and n are finite, one describes with these conditions a finite groupoid, and an infinite groupoid if m or/and n are not finite.

The laws of internal composition and the groupoids thus defined will be called laws of fuzzy internal composition or fuzzy internal laws and fuzzy groupoids.

Example :

Let E={A, B} and M={0, ½, 1}.

$\underset{\sim}{P}(E) =$ {{(A|0), (B|0)}, {(A|0), (B|1/2)}, {(A|1/2), (B|0)}, {(A|1/2), (B|1/2)}, {(A|0), (B|1)}, {(A|1), (B|0)}, {(A|1/2), (B|1)}, {(A|1), (B|1/2)}, {(A|1), (B|1)}}

Designate, in order to simplify writing, for $\underset{\sim}{X} \subset E$,

$$\left\{\left(A \middle| \mu_{\underset{\sim}{X}}(A)\right), \left(B \middle| \mu_{\underset{\sim}{X}}(B)\right)\right\}$$

by

$$(\mu_{\underset{\sim}{X}}(A), \mu_{\underset{\sim}{X}}(B))$$

Thus $\left\{\left(A \middle| \frac{1}{2}\right), (B|0)\right\}$ will be designated by (1/2, 0). With this notation, the table of figure 45.1 represents a fuzzy groupoid.

$\mathfrak{P}(E)$ \ $\mathfrak{P}(E)$	$(0,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	$(1,0)$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(1,1)$
$(0,0)$	$(0,1)$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(1,\frac{1}{2})$	$(0,1)$	$(0,1)$	$(1,\frac{1}{2})$	$(1,1)$	$(1,\frac{1}{2})$
$(0,\frac{1}{2})$	$(1,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},1)$	$(0,0)$	$(1,0)$	$(0,1)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(\frac{1}{2},0)$	$(0,0)$	$(1,\frac{1}{2})$	$(0,\frac{1}{2})$	$(0,1)$	$(1,\frac{1}{2})$	$(1,1)$	$(1,1)$	$(1,1)$	$(\frac{1}{2},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{2})$	$(1,1)$	$(0,0)$	$(1,1)$	$(\frac{1}{2},1)$	$(1,0)$	$(1,0)$	$(\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2})$	$(0,0)$
$(0,1)$	$(\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	$(1,0)$	$(\frac{1}{2},0)$	$(1,\frac{1}{2})$	$(\frac{1}{2},1)$
$(1,0)$	$(\frac{1}{2},0)$	$(0,0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	$(1,0)$	$(0,1)$	$(1,1)$	$(1,\frac{1}{2})$
$(\frac{1}{2},1)$	$(0,0)$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	$(0,\frac{1}{2})$	$(1,0)$	$(1,0)$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(1,1)$
$(1,\frac{1}{2})$	$(0,0)$	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$	$(0,0)$	$(1,\frac{1}{2})$	$(\frac{1}{2},1)$	$(0,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$
$(1,1)$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$

Figure 45.1

Example 2:

If the operation * being considered is intersection $\cap$ and if $\underset{\smile}{A} \subset E$ and $\underset{\smile}{B} \subset E$, one may form a groupoid with the fuzzy subsets $\underset{\smile}{A} \cap \underset{\smile}{B}$.

CONSTRUCTION OF A FUZZY GROUPOID

It suffices to be given a reference set E, finite or not, to deduce $\underset{\smile}{P}\,(E)$ explicitly or not, and to define a law * that corresponds to each ordered pair of fuzzy subsets $(\underset{\smile}{A}, \underset{\smile}{B})$ one and only one fuzzy subset $\underset{\smile}{C}\left(\underset{\smile}{A}, \underset{\smile}{B}, \underset{\smile}{C} \subset E\right)$.

Example 1:

Consider 45.1 and 45.2 again with the law

$$\underset{\smile}{A} * \underset{\smile}{B} = \underset{\smile}{A} \cap \underset{\smile}{B}$$

that is,

$$\mu_{\underset{\smile}{A} \cap \underset{\smile}{B}}(x) = MIN\left(\mu_{\underset{\smile}{A}}(x), \mu_{\underset{\smile}{B}}(x)\right) = \mu_{\underset{\smile}{A}}(x) \wedge \mu_{\underset{\smile}{B}}(x)$$

The constructed groupoid is represented in Figure 45.2.

$\underset{\smile}{P}(E)$ \ $\underset{\smile}{P}(E)$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(0, 1)$	$(1, 0)$	$(\frac{1}{2}, 1)$	$(1, \frac{1}{2})$	$(1, 1)$
$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
$(0, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$	$(0, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$	$(0, \frac{1}{2})$	$(0, \frac{1}{2})$
$(\frac{1}{2}, 0)$	$(0, 0)$	$(0, 0)$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$	$(0, 0)$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$
$(\frac{1}{2}, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$(0, 1)$	$(0, 0)$	$(0, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$	$(0, 1)$	$(0, 0)$	$(0, 1)$	$(0, \frac{1}{2})$	$(0, 1)$
$(1, 0)$	$(0, 0)$	$(0, 0)$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$	$(0, 0)$	$(1, 0)$	$(\frac{1}{2}, 0)$	$(1, 0)$	$(1, 0)$
$(\frac{1}{2}, 1)$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(0, 1)$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, 1)$
$(1, \frac{1}{2})$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(0, \frac{1}{2})$	$(1, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(1, \frac{1}{2})$	$(1, \frac{1}{2})$
$(1, 1)$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(0, 1)$	$(1, 0)$	$(\frac{1}{2}, 1)$	$(1, \frac{1}{2})$	$(1, 1)$

Figure 45.2

Example 2:

We attempt to define the "fuzzy positive integers". We begin by defining a fuzzy number $\underset{\smile}{1}$ with a membership function $\mu_{\underset{\smile}{1}}(\eta)$, arbitrary but such that

$$\sum_{\eta=0}^{\infty} \mu_{\underbrace{1}}(\eta) = 1, n = 0,1,2,3 \ldots \ldots \ldots \ldots$$

For example,

$$\underbrace{1} = \{(0|0,1), (1|0,8), (2|0,1) \ldots \ldots \ldots . (N > 2|0)\}$$

We form $\underbrace{2}$ in the following fashion.

$$\mu_{\underbrace{2}}(0) = \mu_{\underbrace{1}}(0).\mu_{\underbrace{1}}(0) = (0,1).(0,1) = 0.01$$

$$\mu_{\underbrace{2}}(1) = \mu_{\underbrace{1}}(0).\mu_{\underbrace{1}}(1) + \mu_{\underbrace{1}}(1).\mu_{\underbrace{1}}(0) = (0,1).(0,8) + (0,8).(0,1) = 0.16$$

$$\mu_{\underbrace{2}}(2) = \mu_{\underbrace{1}}(0).\mu_{\underbrace{1}}(2) + \mu_{\underbrace{1}}(1).\mu_{\underbrace{1}}(1) + \mu_{\underbrace{1}}(2).\mu_{\underbrace{1}}(0)$$

$$= (0,1).(0,1) + (0,8).(0,8) + (0,1).(0,1) = 0.66$$

$$\mu_{\underbrace{2}}(3) = \mu_{\underbrace{1}}(1).\mu_{\underbrace{1}}(2) + \mu_{\underbrace{1}}(2).\mu_{\underbrace{1}}(1) = (0,8).(0,1) + (0,1).(0,8) = 0.16$$

$$\mu_{\underbrace{2}}(4) = \mu_{\underbrace{1}}(2).\mu_{\underbrace{1}}(2) = (0,1).(0,1) = 0.01$$

$$\mu_{\underbrace{2}}(N > 4) = 0$$

Thus $\underbrace{2} = \{(0|0.01), (1|0.16), (2|0.66), (3|0.16), (4|0.01), \ldots \ldots, (N > 4|0)\}$

Thus we generalize the formula that

$$\mu_{\underbrace{A}*\underbrace{B}}(N) = \sum_{r=0}^{N} \mu_{\underbrace{A}}(r).\mu_{\underbrace{B}}(N-r) = \sum_{r=0}^{N} \mu_{\underbrace{B}}(r).\mu_{\underbrace{A}}(N-r)$$

For $\underbrace{3}$

$$\mu_{\underbrace{3}}(N) = \mu_{\underbrace{2}*\underbrace{1}}(N) = \sum_{r=0}^{N} \mu_{\underbrace{2}}(r).\mu_{\underbrace{1}}(N-r), N \leq 6.$$

Thus, $\underbrace{3} = \left\{ \begin{matrix} (0|0.001), (1|0.024), (2|0.195), (3|0.560), \\ (4|0.195), (5|0.024), (6|0.001) \ldots \ldots . . (N > 6|0) \end{matrix} \right\}$

and thus it goes.

PROPERTIES OF GROUPOID

(i) Associativity : $\left(\underbrace{A} * \underbrace{B}\right) * \underbrace{C} = \underbrace{A} * (\underbrace{B} * \underbrace{C})$

(ii) Commutativity : $\underbrace{A} * \underbrace{B} = \underbrace{B} * \underbrace{A}$

Example 3:

We consider two fuzzy subsets $\underbrace{A} \subset R$ and $\underbrace{B} \subset R$ with which we produce other fuzzy subsets. Let

$$\mu_{\underbrace{A}}(x) = \frac{1}{\sqrt{2\pi{\sigma_1}^2}} e^{-\frac{(x-a)^2}{2{\sigma_1}^2}}, \quad a, \sigma_1 \in R^*$$

$$\mu_{\underset{\smile}{B}}(x) = \frac{1}{\sqrt{2\pi{\sigma_2}^2}} e^{-\frac{(x-b)^2}{2{\sigma_2}^2}}, \qquad b, \sigma_2 \in R^*$$

One then considers the composition product

$$\mu_{\underset{\smile}{A}*\underset{\smile}{B}}(x) = \int_{-\infty}^{\infty} \mu_{\underset{\smile}{A}}(t)\mu_{\underset{\smile}{B}}(x-t)dt$$

$$= \int_{-\infty}^{\infty} \mu_{\underset{\smile}{B}}(t)\mu_{\underset{\smile}{A}}(x-t)dt$$

$$= \frac{1}{\sqrt{2\pi({\sigma_1}^2 + {\sigma_2}^2)}} e^{-\frac{(x-a-b)^2}{2({\sigma_1}^2+{\sigma_2}^2)}}$$

This permits one to define the fuzzy number $\underset{\smile}{A} * \underset{\smile}{B}$.

In the same manner one generates other fuzzy numbers.

$$\underset{\smile}{A} * \underset{\smile}{A}, \underset{\smile}{B} * \underset{\smile}{B}, \underset{\smile}{A} * \underset{\smile}{A} * \underset{\smile}{A}, \underset{\smile}{A} * \underset{\smile}{A} * \underset{\smile}{B}, \dots\dots. \underset{\smile}{A}^r * \underset{\smile}{B}^s, \dots\dots\dots$$

where the superscripts indicate that there are r-1 compositions of $\underset{\smile}{A}$ and s-1 compositions of $\underset{\smile}{B}$.

With the two fuzzy numbers $\underset{\smile}{A}$ and $\underset{\smile}{B}$ one then generates

$$\underset{\smile}{A}, \underset{\smile}{B}, \underset{\smile}{A} * \underset{\smile}{A}, \underset{\smile}{A} * \underset{\smile}{B}, \underset{\smile}{B} * \underset{\smile}{B}, \dots\dots.. \underset{\smile}{A}^r * \underset{\smile}{B}^s, \dots\dots\dots$$

and the set

$$Q = \left\{\underset{\smile}{A}, \underset{\smile}{B}, \underset{\smile}{A} * \underset{\smile}{A}, \underset{\smile}{A} * \underset{\smile}{B}, \underset{\smile}{B} * \underset{\smile}{B}, \dots\dots.. \underset{\smile}{A}^r * \underset{\smile}{B}^s, \dots\dots\dots.\right\}$$

has the structure of a groupoid, which is moreover associative an commutative.

PRINCIPAL PROPERTIES CONCERNING FUZZY GROUPOIDS

Let * be a law of internal composition of a fuzzy groupoid, we define several properties. This groupoid will be designated by $\left(\underset{\smile}{P}(E), *\right)$.

Commutativity

If, for all ordered pairs $(\underset{\smile}{A}, \underset{\smile}{B}) \in \underset{\smile}{P}(E) \times \underset{\smile}{P}(E)$, $\underset{\smile}{A} * \underset{\smile}{B} = \underset{\smile}{B} * \underset{\smile}{A}$

The law of internal composition is commutative. Also the groupoid is commutative. Thus, for example, the groupoid of Fig. 45.2 is commutative whereas of Fig. 45.1 is not.

For example, in Fig. 45.2, we may verify

$$\left\{\left(A\middle|\tfrac{1}{2}\right), (B|1)\right\} \wedge \{(A|1), (B|0)\} = \left\{\left(A\middle|\tfrac{1}{2}\right), (B|0)\right\}.$$

$$\{(A|1),(B|0)\}\wedge\left\{\left(A\middle|\tfrac{1}{2}\right),(B|1)\right\}=\left\{\left(A\middle|\tfrac{1}{2}\right),(B|0)\right\}.$$

Being given the definition of the law * for fuzzy subsets, one may thence

$$\mu_{\underset{\smile}{A}*\underset{\smile}{B}}(x)=\mu_{\underset{\smile}{A}}(x)\circledast\mu_{\underset{\smile}{B}}(x).$$

commutativity for $\circledast$ implies commutativity for *, and vice versa.

Associativity:

If $\forall\ \underset{\smile}{A},\underset{\smile}{B},\underset{\smile}{C}\subset E$: $\left(\underset{\smile}{A}*\ \underset{\smile}{B}\right)*\ \underset{\smile}{C}=\underset{\smile}{A}*\left(\underset{\smile}{B}*\ \underset{\smile}{C}\right)$,

one says that the law is associative, one also says that the groupoid is associative.

Thus, the groupoid of Fig. 45.2 is associative, whereas that of Fig. 45.1 is not. Thus in Fig. 45.2 one may verify, using the abbreviated notation,

$$\left(\frac{1}{2},\frac{1}{2}\right)\wedge(1{,}0)\wedge\left(\frac{1}{2},1\right)=\left(\frac{1}{2},0\right)\wedge\left(\frac{1}{2},1\right)=\left(\frac{1}{2},0\right),$$

$\left(\frac{1}{2},\frac{1}{2}\right)\wedge(1{,}0)\wedge\left(\frac{1}{2},1\right)=\left(\frac{1}{2},\frac{1}{2}\right)\wedge\left(\frac{1}{2},0\right)=\left(\frac{1}{2},0\right).$

Being given the definition of the law * for the fuzzy subsets, we have,

$$\left(\mu_{\underset{\smile}{A}}(x)\circledast\mu_{\underset{\smile}{B}}(x)\right)\circledast\mu_{\underset{\smile}{C}}(x)=\mu_{\underset{\smile}{A}}(x)\circledast(\mu_{\underset{\smile}{B}}(x)\circledast\mu_{\underset{\smile}{C}}(x)).$$

associativity for $\circledast$ implies associativity for *, and vice versa.

Identity element

An element $e\in E,$ if it exists, such that

$$\forall\ a\in E{:}\,e*a=a.$$

This element e is called a left identity.

In the same manner, an element $e'\in E,$ if it exists, such that

$$\forall\ a\in E{:}\,a*e'=a.$$

This element e' is called a right identity.

An element that is both a left identity and a right identity is called an identity.

When an identity element exists, it is always unique. In fact, if there exist another such element e, we have

$$e*e=e*e=e$$

In a fuzzy groupoid one may define an identity in the same manner. We take the example of Figure 45.2, it is evident that (1,1) is a left identity and a right identity, that is, an identity. In fact $\forall x\in\left\{0,\frac{1}{2},1\right\}$ and $\forall y\in\left\{0,\frac{1}{2},1\right\}$.

$$(1{,}1)\wedge(x,y)=(x,y)\wedge(1{,}1)=(x,y).$$

We shall say that a fuzzy groupoid possesses a left identity $\underset{\smile}{U}$ for the law * if

$\forall \underset{\smile}{A} \subset E\colon \underset{\smile}{U} * \underset{\smile}{A} = \underset{\smile}{A}$

and possesses a right identity $\underset{\smile}{U'}$ for this law if,

$$\forall \underset{\smile}{A} \subset E\colon \underset{\smile}{A} * \underset{\smile}{U'} = \underset{\smile}{A}$$

and possess a unique identity if

$$\forall \underset{\smile}{A} \subset E\colon \underset{\smile}{U} * \underset{\smile}{A} = \underset{\smile}{A} * \underset{\smile}{U} = \underset{\smile}{A}.$$

We have seen with the example of Fig. 45.2 the case of a fuzzy groupoid that possesses an identity.

Inverse element

We consider a law for which there exists an identity e. Then let there be two elements $a\ and\ \bar{a} \in E$. If

$$\bar{a} * a = e$$

then $\bar{a}$ is the left inverse of a. In the same manner, if

$$a * \overline{a'} = e$$

one says that $\overline{a'}$ is the right inverse of a. Finally if $\overline{a'} = a, then$

$$\bar{a} * a = a * \bar{a} = e.$$

then $\bar{a}$ is the inverse of a.

In a fuzzy groupoid we define an inverse for each element.

Again we take the example of Fig. 45.2, We have seen that there exists an identity, which is (1,1). It is clear that there is only one element that composed with itself is able to give (1,1).

For any others such that $(a, b) \prec (1,1) and\ (a', b') \prec (1,1)$, such that

$$(a, b) \wedge (a', b') \prec (1,1).$$

Thus the groupoid of Fig. 45.2 does not have the property of possessing an inverse for each of its elements.

More generally, if the law * is $\cup\ or\ \cap$, one cannot have an inverse. In the case of $\cup$, there is an identity that is defined by : $\forall x \in E\colon \mu_{\underset{\smile}{A}}(x) = 0$; and in the case of $\cap$, there is an identity defined by: $\forall x \in E\colon \mu_{\underset{\smile}{A}}(x) = 1$. But for neither of these cases can one define an inverse, no matter what the fuzzy subset.

$$\forall x \in E\colon \mu_{\underset{\smile}{A}}(x) = 0 \Leftrightarrow \underset{\smile}{A} = \emptyset.$$

$$\forall x \in E\colon \mu_{\underset{\smile}{A}}(x) = 1 \Leftrightarrow \underset{\smile}{A} = E.$$

But, if $\emptyset$ is the identity for $\cup$ and E is the identity for $\cap$, these do not allow one to define inverses; any element such as $\underbrace{B}$ may not give

$$\underbrace{A} \cup \underbrace{B} = \emptyset, unless\ \underbrace{A} = \emptyset\ and\ \underbrace{B} = \emptyset$$

$$\underbrace{A} \cup \underbrace{B} = E, unless\ \underbrace{A} = E\ and\ \underbrace{B} = E.$$

Distributivity

Let * and $*'$ represent two laws of internal composition on the same set E.

If $\forall\ \underbrace{A}, \underbrace{B}, \underbrace{C} \subset E$: $\underbrace{A} * \left(\underbrace{B} *' \ \underbrace{C}\right) = \left(\underbrace{A} * \underbrace{B}\right) *' \left(\underbrace{A} * \ \underbrace{C}\right)$

We say that * is left distributive with respect to the law $*'$.

Likewise, if $\forall\ \underbrace{A}, \underbrace{B}, \underbrace{C} \subset E$: $\underbrace{A} *' \left(\underbrace{B} * \underbrace{C}\right) = \left(\underbrace{A} *' \underbrace{B}\right) * \left(\underbrace{A} *' \ \underbrace{C}\right)$

We say that $*'$ is right distributive with respect to the law *.

A law * left and right distributive with respect to another law $*'$ is said to be distributive with respect to $*'$. Then we write

$$\left(\underbrace{A} *' \underbrace{B}\right) * \left(\underbrace{C} *' \underbrace{D}\right) = \left(\underbrace{A} * \ \underbrace{C}\right) *' \left(\underbrace{A} * \ \underbrace{D}\right) *' \left(\underbrace{B} * \underbrace{C}\right) *' \left(\underbrace{B} * \underbrace{D}\right).$$

Consider, for example, the groupoid already presented in Fig. 45.2. We may verity that

$\Delta_1 = \left\{(0,0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right)\right\}$ is closed.

$\Delta_2 = \left\{\left(\frac{1}{2}, 1\right), \left(1, \frac{1}{2}\right)\right\}$ is not closed.

$\underset{\sim}{\wp}(E)$ \ $\underset{\sim}{\wp}(E)$	$(0,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	$(1,0)$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	$(1,0)$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(1,1)$
$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	$(1,\frac{1}{2})$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(1,1)$
$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$	$(1,0)$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(1,1)$
$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(0,1)$	$(1,1)$	$(\frac{1}{2},1)$	$(1,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(1,\frac{1}{2})$	$(1,0)$	$(1,\frac{1}{2})$	$(1,1)$	$(1,0)$	$(1,1)$	$(1,\frac{1}{2})$	$(1,1)$
$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(1,1)$	$(\frac{1}{2},1)$	$(1,1)$	$(1,1)$
$(1,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,1)$	$(1,\frac{1}{2})$	$(1,1)$	$(1,\frac{1}{2})$	$(1,1)$
$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$

Figure 46.1

In Fig. 46.1 we have represented the same groupoid as that in Fig. 45.2 but with the law $\cup$:

$\Delta_1 = \left\{(0,0), \left(0,\frac{1}{2}\right), \left(\frac{1}{2},0\right)\right\}$ is not closed.

$\Delta_2 = \left\{\left(\frac{1}{2},1\right), \left(1,\frac{1}{2}\right)\right\}$ is not closed.

$\Delta_3 = \left\{\left(\frac{1}{2},1\right), \left(1,\frac{1}{2}\right), (1,1)\right\}$ is closed.

It is interesting to show how to obtain closed subsets for the examples of Figures 45.2 and 46.1 with the aid of a Hasse diagram of the vector lattice representing $\underset{\sim}{P}(E)$, See Figure 46.2.

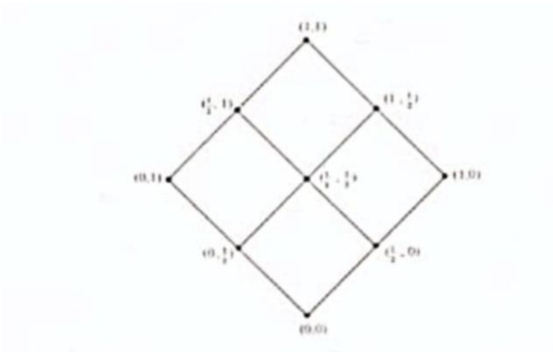


Figure 46.2

The rule is as follows: For the operation ∩, any subset of $\underset{\smile}{P}(E)$, in order to be closed must contain the inferior limit of any pair $(\underset{\smile}{A},\underset{\smile}{B}), \underset{\smile}{A},\underset{\smile}{B} \in \Delta$. Thus, $\left\{(0,0),\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right),(1,0)\right\}$ is closed for ∩. On the other hand, $\left\{\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right),\left(\frac{1}{2},1\right),\left(1,\frac{1}{2}\right)\right\}$ is not closed for ∩. For the operation ∪, but considers superior limits. Thus $\left\{(0,0),\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right),(1,0)\right\}$ is not closed for ∪, whereas $\left\{\left(0,\frac{1}{2}\right),\left(\frac{1}{2},0\right),\left(\frac{1}{2},1\right),\left(1,\frac{1}{2}\right)\right\} \text{ is closed for } \cup$.

Subgroupoids

Any subset $\Delta \subset \underset{\smile}{P}(E)$ closed for a law * will be called a subgroupoid of $(E,*)$ and be denoted $(\Delta \subset E,*)$ or $(\Delta,*)$.

FUZZY MONOIDS

Any fuzzy groupoid that is associative and has an identity will be called fuzzy monoid.

If a monoid possesses in addition the property of commutativity, one calls it a commutative monoid.

All the following fuzzy groupoids, defined by their membership functions and the internal law specified and indicated below, are monoids that are, moreover, commutative.

$(\underset{\smile}{P}(E),\cap)$ where $\mu_{\underset{\smile}{A}\cap\underset{\smile}{B}}(x) = \mu_{\underset{\smile}{A}}(x) \wedge \mu_{\underset{\smile}{B}}(x), \underset{\smile}{A},\underset{\smile}{B} \subset E$.

Associativity is evident. The identity is the reference set E,

$(\underset{\smile}{P}(E),\cup)$ where $\mu_{\underset{\smile}{A}\cup\underset{\smile}{B}}(x) = \mu_{\underset{\smile}{A}}(x) \vee \mu_{\underset{\smile}{B}}(x), \underset{\smile}{A},\underset{\smile}{B} \subset E$.

Associativity is evident. The identity is φ.

$(\underset{\smile}{P}(E),\cdot)$ where $\mu_{\underset{\smile}{A}\cdot\underset{\smile}{B}}(x) = \mu_{\underset{\smile}{A}}(x) \cdot \mu_{\underset{\smile}{B}}(x), \ \underset{\smile}{A},\underset{\smile}{B} \subset E$.

This is associative, with identity E.

$(\underset{\smile}{P}(E),\widetilde{+})$ where $\mu_{\underset{\smile}{A}\,\widetilde{+}\,\underset{\smile}{B}}(x) = \mu_{\underset{\smile}{A}}(x) + \mu_{\underset{\smile}{B}}(x) - \mu_{\underset{\smile}{A}}(x)\,\mu_{\underset{\smile}{B}}(x), \ \underset{\smile}{A},\underset{\smile}{B} \subset E$.

Associative, with identity φ.

$(\underset{\smile}{P}(E),\varphi)$ where $\mu_{\underset{\smile}{A}\,\varphi\,\underset{\smile}{B}}(x) = [\mu_{\underset{\smile}{A}}(x)\wedge\left(1-\mu_{\underset{\smile}{B}}(x)\right)] \vee \left[\mu_{\underset{\smile}{B}}(x)\wedge\left(1-\mu_{\underset{\smile}{A}}(x)\right)\right], \ \underset{\smile}{A},\underset{\smile}{B} \subset E$.

Associative, with identity φ.

A fuzzy monoid will be denoted by $(E,*)$ or preferable, $\left(\underbrace{P}(E),*\right)$.

We shall see several fuzzy groupoids that are not monoids.

Example 1 :

Let $\underbrace{A}, \underbrace{B}$ *be such that* $\mu_{\underbrace{A}\circ\underbrace{B}}(x) = \left|\mu_{\underbrace{A}}(x) - \mu_{\underbrace{B}}(x)\right|$.

Put $a = \mu_{\underbrace{A}}(x)$, $b = \mu_{\underbrace{B}}(x)$, $c = \mu_{\underbrace{C}}(x)$.

and denote $a \odot b = |a - b|$.

It is easy to show that

$$(a \odot b) \odot c \neq a \odot (b \odot c)$$

that is, $\left||a-b| - c\right| \neq \left|a - |b-c|\right|$.

For example, if $a = 0.3$, $b = 0.5$, $c = 0.9$.

$$\left||a-b| - c\right| = \left||0.3 - 0.5| - 0.9\right|$$

$$= |0.2 - 0.9| = 0.7$$

$$\left|a - |b-c|\right| = \left|0.3 - |0.5 - 0.9|\right|$$

$$= |0.3 - 0.4| = 0.1.$$

This commutative groupoid is not a monoid since it is not associative.

Example 2 :

Let $\underbrace{A}, \underbrace{B}$ *be such that* $\mu_{\underbrace{A}\circ\underbrace{B}}(x) = \left|\mu_{\underbrace{A}}(x) - \mu_{\underbrace{B}}(x)\right|$.

Put $a = \mu_{\underbrace{A}}(x)$, $b = \mu_{\underbrace{B}}(x)$, $c = \mu_{\underbrace{C}}(x)$.

and denote $a \odot b = a + kb - ab, k \in [0,1]$.

Now, $(a \odot b) \odot c = (a + kb - ab) \odot c$

$$= (a + kb - ab) + k(a + kb - ab)c - (a + kb - ab)c$$

$$= a + kb + kc - ab - ac - kbc + abc.$$

$$a \odot (b \odot c) = a \odot (b + kc - bc)$$

$$= a + k(b + kc - bc) - a(b + kc - bc)$$

$$= a + kb + k^2c - ab - kac - kbc + abc.$$

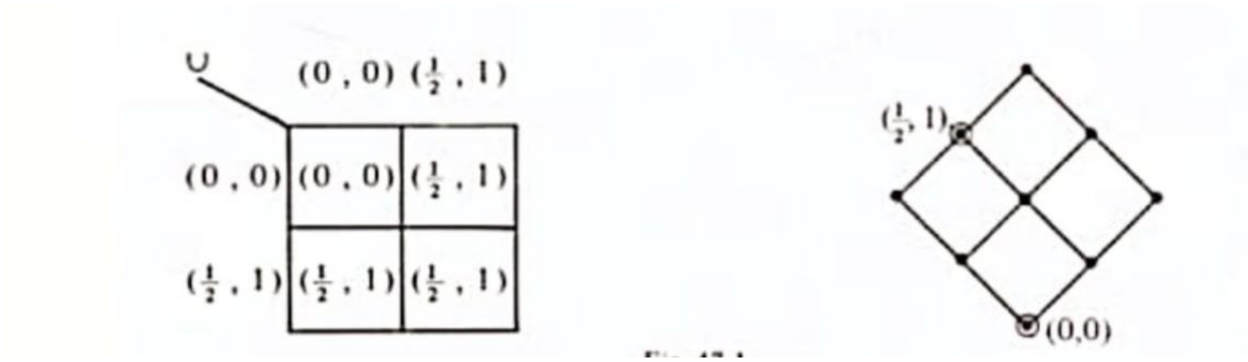
$$[(a \circledcirc b) \circledcirc c] - [a \circledcirc (b \circledcirc c)] = kc - k^2c - ac + kac$$
$$= c(1-k)(k-a).$$

Thus, associativity does not hold, unless k=1.

Fuzzy submonoid:

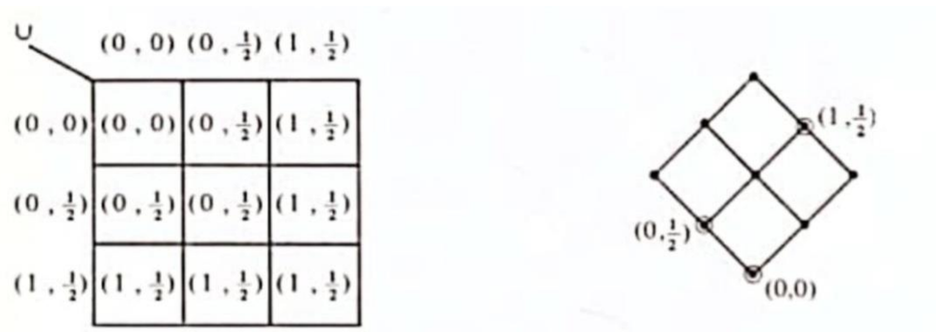
Let $(\underset{\smile}{P}(E), \circ)$ be a fuzzy monoid and $\Delta \subset \underset{\smile}{P}(E)$ be closed for the law $\circ$ then Δ will be called a fuzzy submonoid of $(\underset{\smile}{P}(E), \circ)$ and will be designated by $(\Delta, \circ)$.

Example : Consider the monoid $(\underset{\smile}{P}(E), \cup)$ represented in Figure 46.1.



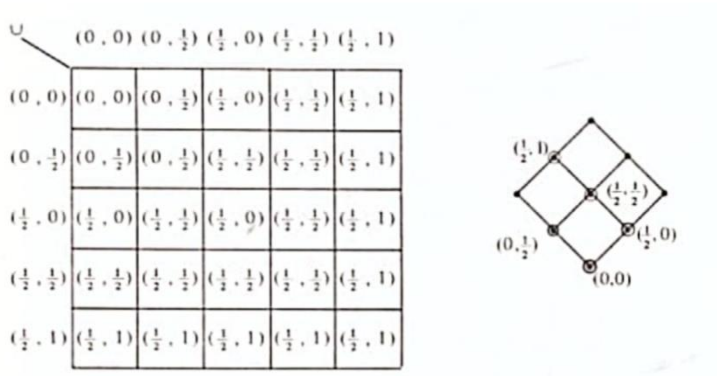
$\cup$	$(0,0)$	$(\frac{1}{2},1)$
$(0,0)$	$(0,0)$	$(\frac{1}{2},1)$
$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$

Figure 47.1



$\cup$	$(0,0)$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$
$(0,0)$	$(0,0)$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$
$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$
$(1,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{1}{2})$

Figure 47.2



$\cup$	$(0,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(0,0)$	$(0,0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$	$(\frac{1}{2},1)$

Figure 47.3

Figure 47.1-47.3 represent submonoids of this monoid.

$$\Delta = \left\{(0,0), \left(\frac{1}{2}, 1\right)\right\}$$

$$\Delta' = \left\{(0,0), \left(0, \frac{1}{2}\right), \left(1, \frac{1}{2}\right)\right\}$$

$$\Delta'' = \left\{(0,0), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right)\right\}$$

$$\textit{and there are several others.}$$

Theorem :

If $(\Delta,\circ)$ and $(\Delta',\circ)$ are submonoids of $\left(\underbrace{P}(E),\circ\right)$ then $(\Delta \cap \Delta',\circ)$ is a submonoid of $\left(\underbrace{P}(E),\circ\right)$.

Proof :

That intersection preserves associativity and the identity is evident.

We show then that $\Delta \cap \Delta'$ remains closed for $\circ$.

Let $\underbrace{A}, \underbrace{B} \in \Delta \cap \Delta'$. Then $\underbrace{A} \circ \underbrace{B}$ belongs to Δ by hypothesis.

It also belongs to Δ' by hypothesis. Then $\underbrace{A} \circ \underbrace{B}$ belongs $\Delta \cap \Delta'$ and $\Delta \cap \Delta'$ is closed with respect to $\circ$.

It is not the same for union $\cup$, which does not always preserve the closure property.

Fuzzy groups

A group is a monoid such that each element possesses one and only one inverse.

We shall show that a necessary condition for $\left(\underbrace{P}(E),\circ\right)$ to have a group structure is that M=[0, 1] also have a group structure for an operation corresponding to $\circ$. We shall see that in any case M=[0, 1] may be endowed with a group structure for an operation $\circ$ to be defined.

M=[0, 1] is a vector lattice that may be reduced to a single chain forming a total order. We consider the operations ^(min), $_\vee$ (max), $\circ$ (product), $\mp$ (algebraic sum), $\oplus$ (disjunctive sum). For each of these operations, one has the associative property and there exists an identity, which is, depending on the case 0 or 1; but it is easy to prove, almost in the same way, that for each of these operations, there does not exist an inverse for each element. We show this for A. Consider a pair

$$(a, b) \in M \times M,$$

$M = [0,1]$ and such that $0 < a < b < 1$. The identity of ^ is 1.

$\textit{Does there exist then an } a \textit{ or } b \textit{ such that } a^\wedge b = 1.$

$$This\ is\ impossible\ a^\wedge b = a < 1.$$

On the other hand, if one takes $M = [0,1]$, one finds that a group is possible.

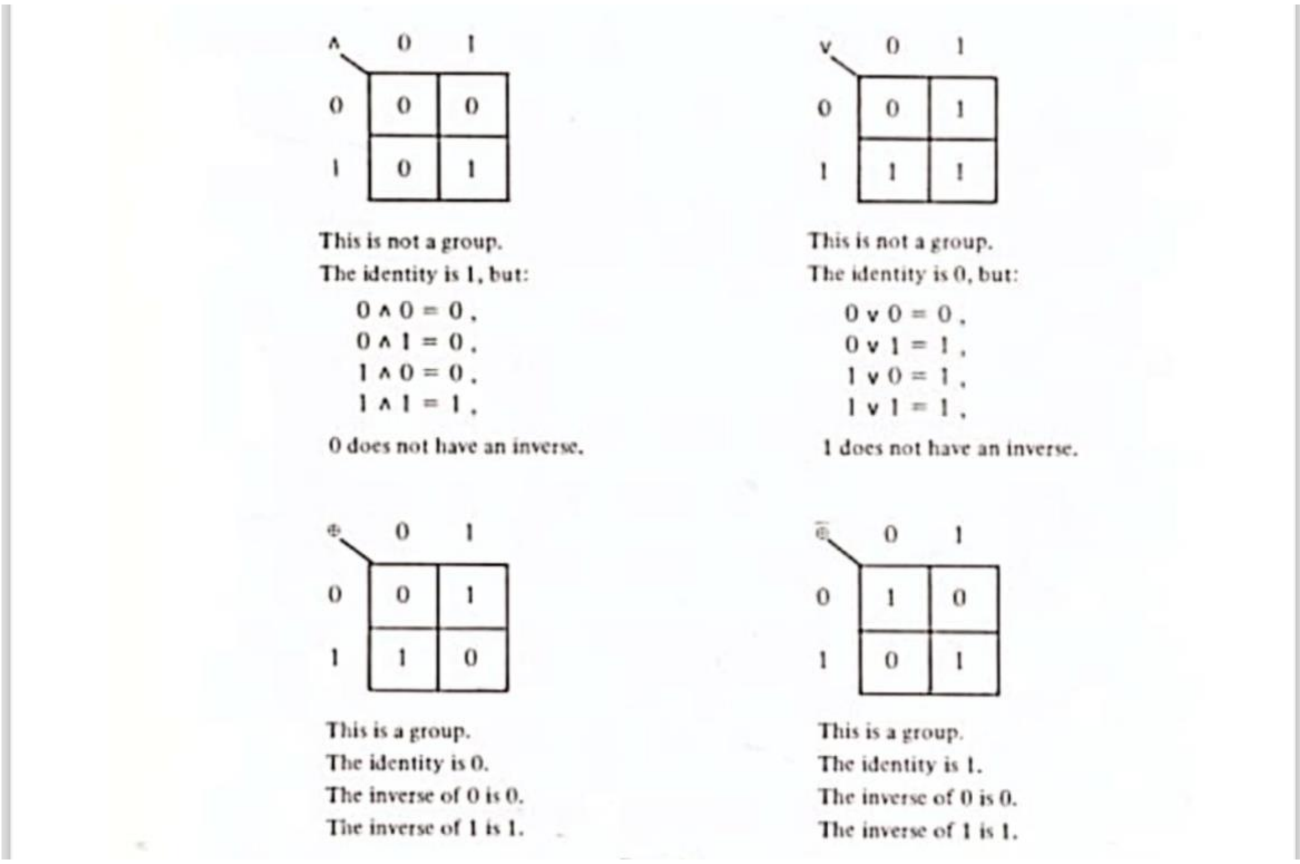


Figure 47.6

Thus, we show in Figure 47.6 that one does not obtain a group for $^\wedge$ or $_\vee$ (one thus does not obtain a group any longer for $\circ$ and $\mp$, which in the Boolean case give equivalent operations). On the contrary, one does obtain a group if one takes the operation $\oplus$. One also obtains a group if one considers the operation $\tilde{\oplus}$ (inverse disjunctive sum). We note that the two groups $\emptyset$ and $\tilde{\emptyset}$ are isomorphic by permitting 0 and 1; a single group may represent the two.

It follows from this that if one considers any one of the operations $\cup, \cap, \circ, \mp, \oplus$ and M=[0,1], one may not give $\left(\underset{\smile}{P}\,(E), \circ\right)$ a group structure.

If one takes M=[0,1], it is only with $\oplus$ (or what amounts to the same thing, with $\tilde{\oplus}$) that one may form a group. We consider as an example the ordinary group formed thus with E=$\{x_1, x_2, x_3\}$.

⊕	000	001	010	011	100	101	110	111
000	000	001	010	011	100	101	110	111
001	001	000	011	010	101	100	111	110
010	010	011	000	001	110	111	100	101
011	011	010	001	000	111	110	101	100
100	100	101	110	111	000	001	010	011
101	101	100	111	110	001	000	011	010
110	110	111	100	101	010	011	000	001
111	111	110	101	100	011	010	001	000

Figure 47.7

If one puts

$$abc = \{(x_1|a), (x_2|b), (x_3|c)\}$$

in order to simplify writing, with

$$a, b, c \in \{0,1\},$$

one obtains the group represented in Figure 47.7. The identity is 000 and each element abc has itself for an inverse. The group $\left(\underset{\sim}{P}(E), \oplus\right)$ has been represented in Figure 47.8 by replacing the binary numbers abc by their corresponding decimals.

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Figure 47.8

FUZZY EXTERNAL COMPOSITION

Let E_1 and E_2 be two sets. If to each ordered pair $\left(\underbrace{A_1}, \underbrace{A_2}\right), \underbrace{A_1} \subset E_1,\ \underbrace{A_2} \subset E_2$, one may correspond one and only one $\underbrace{A_3} \subset E_3$, one has a law of fuzzy external composition if $E_3 \neq E_1\ or\ /and\ \ E_3 \neq E_2$, the law is internal.

We shall consider several examples of laws of fuzzy external composition.

Example 1:

We see first a purely discrete example. Let,

$$E_1 = \{A, B, C\}, \qquad M_1 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\} \ \ ,\ card\ E_1 = 3 \quad, card\ M_1 = 5$$

$$E_2 = \{a, b, c, d\}, \qquad M_2 = \left\{0, \frac{1}{2}, 1\right\} \ \ ,\ card\ E_2 = 4 \quad, card\ M_2 = 3$$

$$E_3 = \{\alpha, \beta\}, \qquad M_3 = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\} \ \ ,\ card\ E_3 = 2 \quad, card\ M_3 = 4.$$

Let $\underbrace{A_1} \subset E_1\ and\ \ \underbrace{A_2} \subset E_2$: to each ordered pair such as $(\underbrace{A_1},\ \underbrace{A_2})$ we correspond one and only one $\underbrace{A_3} \subset E_3$ by means of a table. Thus, let

$$\underbrace{A_1} = \left\{\left(A\middle|\tfrac{1}{4}\right), \left(B\middle|\tfrac{1}{2}\right), (C|1)\right\}\ denoted\ (\frac{1}{4}, \frac{1}{2}, 1)$$

$$\underbrace{A_2} = \left\{(a|0), \left(b\middle|\tfrac{1}{2}\right), (c|0), (d|1)\right\}\ denoted\ \left(0, \frac{1}{2}, 0, 1\right).$$

We suppose that the table corresponds to these two fuzzy subsets

$$\underbrace{A_3} = \left\{\left(\alpha\middle|\tfrac{1}{3}\right), (\beta|1)\right\}\ denoted\ \left(\frac{1}{3}, 1\right)$$

The table will possess $5^3 \times 3^4 = 125 \times 81$ cases: we do not present this, but give a small extract in Figure 48.1

$\mathfrak{L}(E_1)$ \ $\mathfrak{L}(E_2)$		$(0, \frac{1}{2}, 0, \frac{1}{2})$	$(0, \frac{1}{2}, 0, 1)$	$(0, \frac{1}{2}, 1, 0)$	
$(\frac{1}{4}, \frac{1}{2}, 1)$		$(\frac{1}{3}, \frac{2}{3})$	$(\frac{1}{3}, 1)$	$(0, 1)$	
$(\frac{1}{4}, 1, 0)$		$(0, \frac{1}{3})$	$(1, \frac{1}{3})$	$(1, \frac{1}{3})$	
$(\frac{1}{4}, 1, \frac{1}{2})$		$(\frac{2}{3}, 1)$	$(\frac{2}{3}, \frac{2}{3})$	$(\frac{2}{3}, 1)$	
$(\frac{1}{4}, 1, 1)$		$(0, \frac{2}{3})$	$(0, 0)$	$(1, \frac{1}{3})$	

Figure 48.1

Example 2:

We take the same example as above, but with the law,

$$\mu_{\underbrace{A_1}}(\alpha) = \hat{x}\hat{y}|\mu_{\underbrace{A_1}}(x) \vee \mu_{\underbrace{A_2}}(y)|\,,$$

$$\mu_{\underbrace{A_2}}(y) = \overset{\vee}{x}\overset{\vee}{y}|\mu_{\underbrace{A_1}}(x) \wedge \mu_{\underbrace{A_2}}(y)|.$$

one obtain another composition table from which one goes on to calculate an element $\underbrace{P}(E_1) \times \underbrace{P}(E_2)$. Let

$$\underbrace{A_1} = \left\{\left(A\middle|\tfrac{1}{4}\right), \left(B\middle|\tfrac{1}{2}\right), (C|1)\right\} \; denoted \; (\frac{1}{4}, \frac{1}{2}, 1)$$

$$\underbrace{A_2} = \left\{(a|0), \left(b\middle|\tfrac{1}{2}\right), (c|0), (d|1)\right\} \; denoted \; \left(0, \frac{1}{2}, 0, 1\right).$$

$$\mu_{\underbrace{A_3}}(\alpha) = \hat{x} \mid \hat{y} \; \left(\frac{1}{4} \vee 0, \frac{1}{4} \vee \frac{1}{2}, \frac{1}{4} \vee 1\right), \; \hat{y} \; \left(\frac{1}{2} \vee 0, \frac{1}{2} \vee \frac{1}{2}, \frac{1}{2} \vee 0, \frac{1}{2} \vee 1\right),$$

$$\hat{y} \; \left(1 \vee 0\,, 1 \vee \frac{1}{2}\,, 1 \vee 0, 1 \vee 1\right) |$$

$$= \hat{x} \mid \hat{y} \; \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 1\right), \; \hat{y} \; \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right), \hat{y} \; (1\,, 1\,, 1, 1)|$$

$$= \hat{x} \; \left(\frac{1}{4}, \frac{1}{2}, 1\right) = \frac{1}{4}.$$

$$\mu_{\underbrace{A_3}}(\beta) = \overset{\vee}{x} \mid \overset{\vee}{y} \; \left(\frac{1}{4} \wedge 0, \frac{1}{4} \wedge \frac{1}{2}, \frac{1}{4} \wedge 0, \frac{1}{4} \wedge 1\right), \; \overset{\vee}{y} \; \left(\frac{1}{2} \wedge 0, \frac{1}{2} \wedge \frac{1}{2}, \frac{1}{2} \wedge 0, \frac{1}{2} \wedge 1\right),$$

$$\overset{\vee}{y} \; \left(1 \wedge 0\,, 1 \wedge \frac{1}{2}\,, 1 \wedge 0, 1 \wedge 1\right) |$$

$$= \overset{\vee}{x} \mid \overset{\vee}{y} \; \left(0, \frac{1}{4}, 0, \frac{1}{4}\right), \; \overset{\vee}{y} \; \left(0, \frac{1}{2}, 0, \frac{1}{2}\right), \overset{\vee}{y} \; \left(0, \frac{1}{2}, 0, 1\right) |$$

$$= \overset{\vee}{\underset{x}{}}\left(\frac{1}{4}, \frac{1}{2}, 1\right) = 1.$$

Thus, $\mu_{\underbrace{A_3}}(\alpha) = \frac{1}{4} \; and \; \mu_{\underbrace{A_3}}(\beta) = 1.$

To

$$\underbrace{A_1} = \left(A\middle|\tfrac{1}{4}\right), \left(B\middle|\tfrac{1}{2}\right), (C|1)$$

and

$$\underbrace{A}_2 = (a|0), \left(b\middle|\frac{1}{2}\right), (c|0), (d|1)$$

one corresponds

$$\underbrace{A}_3 = \left(\alpha\middle|\frac{1}{4}\right), (\beta|1).$$

Remark :

In the general case, let M_1 be associated with E_1;

M_2 be associated with E_2;

M_3 be associated with E_3.

If $\underbrace{P}(E_3)$ is formed with respect to $\underbrace{P}(E_1)$ by a law $\circ$ corresponding to

$$\mu_{\underbrace{A}_3}(x, y) = \mu_{\underbrace{A}_1}(x) \odot \mu_{\underbrace{A}_2}(y).$$

M_3 will be deduced from M_1 and M_2 by considering the formula of composition.

Thus, in the example of

$$\mu_{\underbrace{A}_1}(\alpha) = \bigwedge_x \bigwedge_y |\mu_{\underbrace{A}_1}(x) \vee \mu_{\underbrace{A}_2}(y)|\ ,$$

$$\mu_{\underbrace{A}_2}(y) = \bigvee_x \bigvee_y |\mu_{\underbrace{A}_1}(x) \wedge \mu_{\underbrace{A}_2}(y)|,$$

it is evident that,

$$M_3 = M_1 \cup M_2 = M_1 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}.$$

Example 3:

We shall construct a fuzzy graph whose vertices are fuzzy subsets, this will define a law of external composition.

Let

$$\underbrace{A} \subset E, \underbrace{B} \subset E,$$

To any ordered pair $(\underbrace{A}, \underbrace{B}) \in \underbrace{P}(E) \times \underbrace{P}(E)$ one will correspond an element denoted $\underbrace{A} \circ \underbrace{B} = c(\underbrace{A} \cdot \underbrace{B})$

The element c takes its values in a set Q defined by the operation $\circ$.

Suppose, for example, that

$$E = \{a, b\} and\ M = \left\{0, \frac{1}{2}, 1\right\}.$$

Suppose also that $c\left(\underset{\smile}{A} \cdot \underset{\smile}{B}\right) = \left[\mu_{\underset{\smile}{A}}(a) \wedge \mu_{\underset{\smile}{B}}(a)\right] \vee \left[\mu_{\underset{\smile}{A}}(b) \wedge \mu_{\underset{\smile}{B}}(b)\right]$.

With such a function, c takes its values in $Q = M = \left\{0, \frac{1}{2}, 1\right\}$.

We obtain the fuzzy graph given in the figure 48.2.

	$(0,0)$	$(0,\frac{1}{2})$	$(0,1)$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$	$(1,0)$	$(1,\frac{1}{2})$	$(1,1)$
$(0,0)$	0	0	0	0	0	0	0	0	0
$(0,\frac{1}{2})$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
$(0,1)$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1
$(\frac{1}{2},0)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\frac{1}{2},\frac{1}{2})$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\frac{1}{2},1)$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1
$(1,0)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1
$(1,\frac{1}{2})$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1
$(1,1)$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1

Fig. 48.2

In this fashion one may construct fuzzy graphs that possess particular properties due to their construction. This is a conception of fuzzy graphs for which the elements of vertices are fuzzy subsets of the same reference set.

It is a matter here of an extension that may have concrete application, for example, if $\circ$ corresponds to an evaluation of distance.

Example 4 :

Recall Example 3 and suppose now that $c(\underset{\smile}{A}, \underset{\smile}{B})$ is the relative generalized Hamming distance given by

$$\delta\left(\underset{\smile}{A}, \underset{\smile}{B}\right) = \frac{1}{2}\left(\left|\mu_{\underset{\smile}{A}}(a) - \mu_{\underset{\smile}{B}}(a)\right| + \left|\mu_{\underset{\smile}{A}}(b) - \mu_{\underset{\smile}{B}}(b)\right|\right).$$

This indeed gives a law of external composition.

$\bullet$	$(0,0)$	$(0,\frac{1}{2})$	$(0,1)$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$	$(1,0)$	$(1,\frac{1}{2})$	$(1,1)$
$(0,0)$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$(0,\frac{1}{2})$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$(0,1)$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$
$(\frac{1}{2},0)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$(\frac{1}{2},\frac{1}{2})$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$
$(\frac{1}{2},1)$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$(1,0)$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
$(1,\frac{1}{2})$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$
$(1,1)$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0

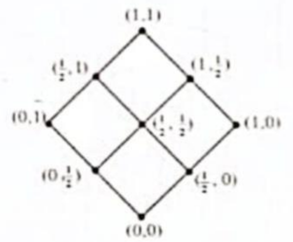
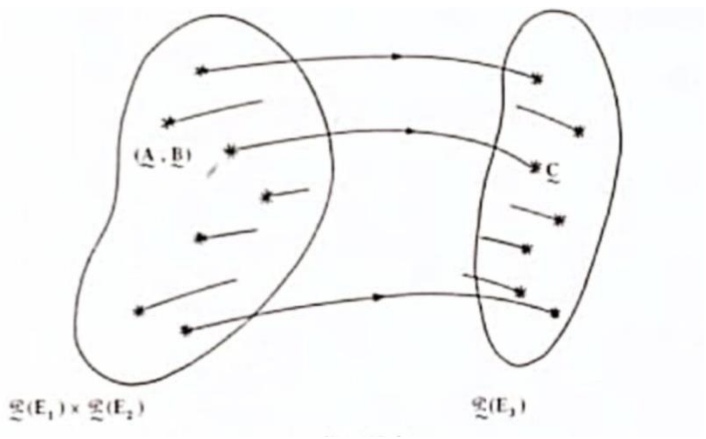


Figure 48.3

Importance of the notion of a law of external composition of fuzzy subsets

This notion is important: it characterizes any system of evaluation of relations among fuzzy subsets of the same reference set, indeed of fuzzy subsets of different reference sets. The set in which $\underset{\sim}{P}\,(E_1) \times \underset{\sim}{P}\,(E_2)$ takes its values may be an ordinary set or more generally a set of fuzzy subsets of an ordinary power set (Figure 48.4).



The notion of distance between messages or fuzzy subsets of the same reference set gives an example (one of the more trivial examples) concerning this general notion.

We remark that the procedures for invention or ingenuity that one calls biassociation are procedures essentially based on laws of external composition. One takes a concept $\underset{\sim}{A}$, which is an ordinary or fuzzy subset of a family of concepts E_1, and another concept $\underset{\sim}{B}$, which is an ordinary or fuzzy subset of another (or eventually the same family. The biassociation of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is an external law $\circ$ that allows one to obtain a new concept $\underset{\sim}{C}$, which is an ordinary or fuzzy subset of a third family (eventually the same as one of the preceding families) (Figure 48.5).

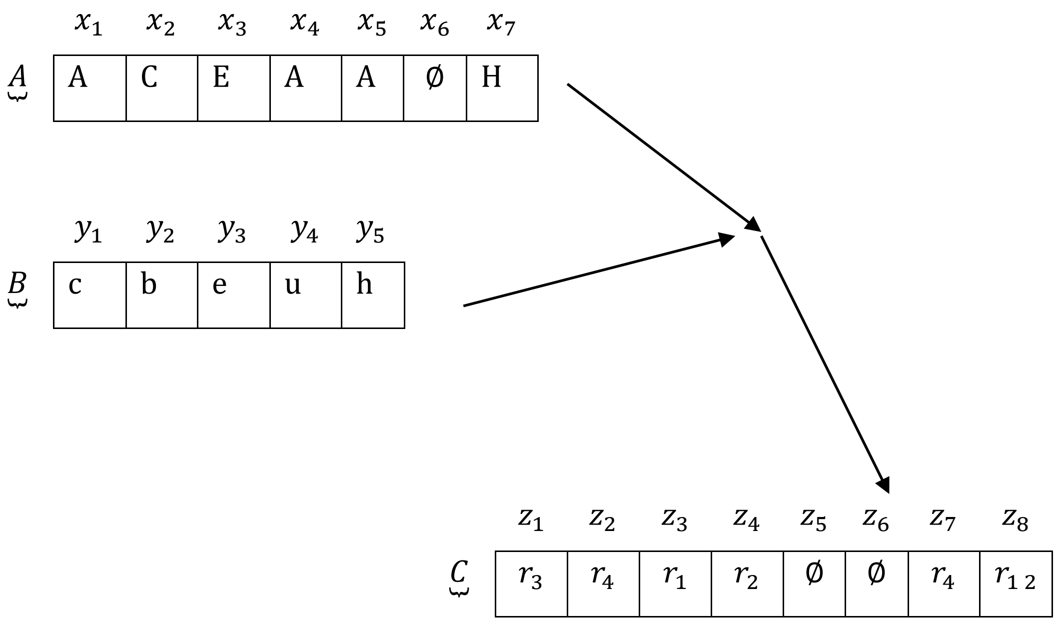


Figure 48.5 The phenomenon of biassociation

OPERATIONS ON FUZZY NUMBERS

We go on to consider various types of fuzzy numbers.

Exponential fuzzy integers

Consider a reference set $E = R^{\circ}$, and the fuzzy subset $\underset{\sim}{I_1}$ such that

$$\mu_{\underset{\sim}{I_1}}(x) = \lambda e^{-\lambda x}, x \in R^{\circ}.$$

Then define $\underset{\sim}{I_2}$ in the following fashion:

$$\mu_{\underbrace{I_2}}(x) = \mu_{\underbrace{I_1}}(x) \circ \mu_{\underbrace{I_1}}(x)$$

$$= \int_0^x \mu_{\underbrace{I_1}}(t)\mu_{\underbrace{I_1}}(x-t)dt$$

$$= \int_0^x \lambda e^{-\lambda t}\lambda e^{-\lambda(x-t)}dt$$

$$= \lambda^2 x e^{-\lambda x}$$

Next define $\underbrace{I_3}$ in the following manner :

$$\mu_{\underbrace{I_3}}(x) = \mu_{\underbrace{I_2}}(x) \circ \mu_{\underbrace{I_1}}(x) = \mu_{\underbrace{I_1}}(x) \circ \mu_{\underbrace{I_2}}(x)$$

$$= \int_0^x \lambda^2 t e^{-\lambda t}\lambda e^{-\lambda(x-t)}dt$$

$$= \frac{\lambda^3 x^2 e^{-\lambda x}}{2}.$$

and $\underbrace{I_n}$:

$$\mu_{\underbrace{I_n}}(x) = \mu_{\underbrace{I_{n-1}}}(x) \circ \mu_{\underbrace{I_1}}(x) = \mu_{\underbrace{I_1}}(x) \circ \mu_{\underbrace{I_{n-1}}}(x)$$

$$= \frac{\lambda^n x^{n-1} e^{-\lambda x}}{n!}.$$

Note that

$$\underset{x}{MAX}\, \mu_{\underbrace{I_n}}(x) = \underset{x}{MAX}\, \frac{\lambda^n x^{n-1} e^{-\lambda x}}{n!}$$

$$= \frac{\lambda(n-1)^{(n-1)} e^{-(n-1)}}{(n-1)!}$$

for the value $x = \frac{n-1}{\lambda}$.

Thus one may establish the values in the following table.

$\underbrace{I_i}$	$\mu_{\underbrace{I_i}}(x)$	Abscissa of the maximum	Ordinate of the maximum
$\underbrace{I_1}$	$\lambda e^{-\lambda x}$	$x = 0$	λ
$\underbrace{I_2}$	$\lambda^2 x e^{-\lambda x}$	$x = \frac{1}{\lambda}$	λe^{-1}
$\underbrace{I_3}$	$\frac{\lambda^3 x^2 e^{-\lambda x}}{2}$	$x = \frac{2}{\lambda}$	$\frac{\lambda 2^2 e^{-2}}{2}$

.... $\underbrace{I_n}$	.... $\frac{\lambda^n x^{n-1} e^{-\lambda x}}{n!}$	.... $x = \frac{n-1}{\lambda}$	.... $\frac{\lambda(n-1)^{(n-1)} e^{-(n-1)}}{(n-1)!}$ ....

The fuzzy subsets

$$\underbrace{I_1}, \quad \underbrace{I_2}, \underbrace{I_3}, \dots \dots \dots \underbrace{I_n}, \dots \dots \dots .$$

will be called exponential fuzzy integers. $\underbrace{I_1}$ will be called exponential fuzzy 1, $\underbrace{I_2}$ will be called exponential fuzzy 2 , etc.

The operation of composition defined above is associative and commutative. Thus the set of fuzzy subsets

$$\underbrace{I_1}, \underbrace{I_2}, \underbrace{I_3}, \dots \dots \dots \underbrace{I_n}, \dots \dots \dots .$$

forms an associative and commutative groupoid.

Geometric fuzzy integers

Consider the reference set

$$E = N$$

and the fuzzy subset $\underbrace{J_1}$ such that

$$\mu_{\underbrace{J_1}}(x) = a(1-a)^{x-1}, a \in R^{\circ}, 0 < |a| < 1, x = 1,2,3, \dots \dots$$

Then we define $\underbrace{J_2}$ in the following form:

$$\mu_{\underbrace{J_2}}(x) = \mu_{\underbrace{J_1}}(x) \circ \mu_{\underbrace{J_1}}(x)$$

$$= \sum_{t=1}^{x-1} \mu_{\underbrace{J_1}}(t) \mu_{\underbrace{J_1}}(x-t)$$

$$= \sum_{t=1}^{x-1} a(1-a)^{t-1} a(1-a)^{x-t-1}$$

$$= a^2(1-a)^{x-2} \sum_{t=1}^{x-1} 1$$

$$= (x-1)a^2(1-a)^{x-2}, x = 2,3,4, \dots \dots \dots ..$$

Then $\underset{\smile}{J_3}$ is defined in the following way:

$$\mu_{\underset{\smile}{J_3}}(x) = \mu_{\underset{\smile}{J_1}}(x) \circ \mu_{\underset{\smile}{J_2}}(x) = \mu_{\underset{\smile}{J_2}}(x) \circ \mu_{\underset{\smile}{J_1}}(x)$$

$$= \sum_{t=2}^{x-1} \mu_{\underset{\smile}{J_1}}(t)\mu_{\underset{\smile}{J_2}}(x-t)$$

$$= \sum_{t=2}^{x-1} (t-1)a^2(1-a)^{t-2}a(1-a)^{x-t-1}$$

$$= a^3(1-a)^{x-3}\sum_{t=2}^{x-1}(t-1)$$

$$= \frac{(x-1)(x-2)}{2}a^3(1-a)^{x-3}, x = 3,4,5, \dots\dots\dots..$$

More generally in the same manner we obtain

$$\mu_{\underset{\smile}{J_r}}(x) = C_{x-1}^{x-r}\, a^r(1-a)^{x-r}.$$

The abscissas of the maximums are x=r, r+1, ... given in the following table.

$\underset{\smile}{J_i}$	$\mu_{\underset{\smile}{J_i}}(x)$	Abscissa of the maximum
$\underset{\smile}{J_1}$	$a(1-a)^{x-1}$	$x = 1$
$\underset{\smile}{J_2}$	$(x-1)a^2(1-a)^{x-2}$	$\frac{1}{a} \leq x \leq 1 + \frac{1}{a}$
$\underset{\smile}{J_3}$	$\frac{(x-1)(x-2)}{2}a^3(1-a)^{x-3}$	$\frac{2}{a} \leq x \leq 1 + 2/a$
....	....	....
....	....	....
....	....	....
$\underset{\smile}{J_r}$	$C_{x-1}^{x-r}\, a^r(1-a)^{x-r}$	$\frac{n-1}{a} \leq x \leq 1 + \frac{n-1}{a}$
....	....	
		....

The fuzzy subsets

$$\underset{\smile}{J_1}, \underset{\smile}{J_2}, \underset{\smile}{J_3}, \dots\dots\dots\dots, \underset{\smile}{J_r}, \dots\dots$$

will be called geometric fuzzy integers. $\underset{\smile}{J_1}$ will be called geometric fuzzy 1, etc.

Gaussian fuzzy integers

Consider a reference set E=R,

and the fuzzy subset $\underbrace{K_1}$ such that

$$\mu_{\underbrace{K_1}}(x) = \frac{1}{\sqrt{2\pi\sigma_1{}^2}} e^{-\frac{(x-1)^2}{2\sigma_1{}^2}}$$

$$\mu_{\underbrace{K_2}}(x) = \mu_{\underbrace{K_1}}(x)^\circ \mu_{\underbrace{K_1}}(x)$$

$$= \int_0^x \mu_{\underbrace{K_1}}(t)\mu_{\underbrace{K_1}}(x-t)dt$$

$$= \frac{1}{\sqrt{4\pi\sigma_1{}^2}} e^{-\frac{(x-2)^2}{4\sigma_1{}^2}}$$

Continuing we get

$$\mu_{\underbrace{K_r}}(x) = \frac{1}{\sqrt{2\pi r\sigma_1{}^2}} e^{-\frac{(x-r)^2}{2r\sigma_1{}^2}}$$

The fuzzy subsets

$$\underbrace{K_1}, \underbrace{K_2}, \underbrace{K_3}, \ldots\ldots\ldots\ldots, \underbrace{K_r}, \ldots..$$

will be called Gaussian fuzzy integers.

CONCLUSION

Fuzzy logic has been used in numerous applications such as facial pattern recognition, air conditioners, washing machines, vacuum cleaners, antiskid braking systems, transmission systems, control of subway systems and unmanned helicopters, knowledge-based systems for multi objective optimization of power systems, ...

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